**PHD**

# Barrier options, time-lagged trading and optimisation

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# Barrier Options, Time-lagged Trading and Optimisation

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# Summary

This thesis is in two distinct parts. Part 1 is mainly taken up with the paper "Fast Accurate Binomial Pricing", which was written jointly with my supervisor Chris Rogers, and published in January 1998. In the paper we develop a new method of binomial tree pricing with random time steps which we show to be particularly suitable for the pricing of barrier options. Results are given for several types of barrier options to test both accuracy and speed in comparison to analytic formulae and previous numerical methods. In this thesis I also provide a chapter of background in barrier option theory and binomial tree methods.

In Part 2 I look at some of the effects of time-lagged trading. It is usually assumed in consumption-investment problems and in hedging strategies that trades can be implemented as soon as the decision to trade is made. However, in reality, there will always be some time lag, whether caused by a lack of available assets or the need to contact traders on the other side of the world. I look at utility maximisation problems under such conditions for investors with two different utility functions. In the first case, for exponential utility, I calculate the fair price of a put option for an investor trading to replicate the payoff. I start by solving the discrete-time problem using binomial tree methods and then go on to continuous time, using a Taylor's expansion. Next, for the power utility function, I look at the effect of the time lag on the expected utility of an investor in discrete time using two distinct methods. Firstly, we obtain a numerical solution using the binomial tree method used for the exponential utility. Secondly, we assume the time-lag is small, ignore terms of high order and approximate the value function at any point by the exponential of a quadratic and thereby obtain a recursive solution. The two methods are compared over several examples.

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# Part I

# Pricing Barrier Options

# Chapter 1

# Introduction

The pricing of barrier options has been widely studied over the last twenty years. As early as 1973, Merton(1973) produced a formula for a standard down and out European call option in his extension of the work of Black and Scholes (1973). Cox and Rubinstein (1985) extended the formula to include the case with rebates and Rubinstein and Reiner (1991) give formulae for all single barrier options (both knock-ins and knock-outs) with and without rebates. In Rich (1994) full derivation is given of all probability densities needed for the pricing of single barrier options. Options with rebates and dividends are dealt with.

More complicated barrier options are also now available (see Rubinstein (1993) for an extensive review of exotic options). A partial barrier option, for example, has a barrier which is only effective over part of the life of the option. Closed form solutions for such options are given in Heynen and Kat (1994b); Hui (1997) which includes the dividend case; and also in Carr (1995) who gives full derivation of the formulae. Hui (1997) also prices the step barrier which has a barrier made up of two or more levels, each effective over a different period. More complex still is the rainbow or outside barrier option. The price of this option is dependent on two different underlying asset prices. One asset price determines the payoff, while a second asset is responsible for determining when, if ever, the barrier is hit. Rainbow barrier options are priced in Carr (1995) and Heynen and Kat (1994a).

Another type of barrier option is the double barrier knock-out option. Various methods have been used to develop an analytic solution. This can be done by calculating the probability density of not hitting either barrier. A derivation of this density is given in Section 1.3.3 of this chapter. However, this method gives the price as an infinite sum of terms and therefore alternatives are still being sought. Geman and Yor (1996) use Laplace transforms. Kunitomo and Ikeda (1992) deal with moving barriers using probabilistic methods and an adaptation of Lèvy's theorem. Bhagavatula and Carr (1995) find solutions for double-barrier options with time-dependent parameters which can be applied to the case of a time-dependent barrier.

The alternative to analytic option pricing is to find a numerical solution. Again extensive work has been done in this area. The basic method for pricing any type of option is the binomial tree developed by Cox, Ross and Rubinstein (1979). A full description of this method is given in Section 1.4. However, as pointed out by Boyle and Lau (1994), the standard binomial tree method can produce a biased price estimate for barrier options. Boyle and Lau demonstrate that although for the standard call option convergence to the true price is fairly quick, for the down-and-out call convergence is slow and even 4000 time steps may not produce sufficient accuracy. Graphs show that as the number of time steps used increases, the estimate approaches the true price at certain points but immediately afterwards the estimate becomes very inaccurate before gradually approaching the true price once more. Boyle and Lau deduce that the true price is approached (for the down-and-out call) when a level of the lattice falls just below the barrier[1]. They make use of this fact by calculating the number of time steps required to ensure that this will occur, thereby minimising the inaccuracy of the method. An alternative method by Derman et al (1995) is also based on the positioning of the lattice relative to the barrier. Rather than moving the lattice however, they move the barrier, calculating the price of an option with barrier on the lattice above the true barrier, and also that with barrier on the lattice below. They then interpolate between the two option prices.

Work has also been done using the trinomial tree, which offers greater flexibility. Ritchken (1995) makes use of this by modifying the trinomial lattice close to the barrier so that it hits it exactly and removes the source of the inaccuracy.

[1] An explanation of this is given in Section 1.4.1

A disadvantage of this method, however, is that the lattice must be calculated specifically for each option. Rogers and Zane (1997) apply the trinomial tree to the problem of continuous moving barriers. They begin by a transformation of the log-price into a process which must stay in the range [0,1] thus removing the problem of time-dependent barriers. A review of numerical methods is given in Broadie and Detemple (1996).

Our method is an alternative binomial tree with random time steps developed to accurately price barrier options. The majority of the material in Chapters 2 and 3 is taken from Rogers and Stapleton (1998) although some extra background and explanation has been added.

In the remainder of this chapter I give an introduction to option theory, barrier options and binomial trees. In Chapter 2, the alternative binomial method is described in full. Results and conclusions can be found in Chapter 3.

## 1.1 Option theory

An option is a financial derivative product traded on the stock exchange. Its value depends on the value of an underlying index, such as the price of a share.

**Definition 1.1.1** *A European call option gives the holder the right, but not the obligation, to buy 1 unit of a given underlying asset for a fixed price $K$ at time $T$.*

*$K$ is called the strike (or exercise) price;*

*$T$ is called the expiry.*

At time $T$ the holder decides either to "exercise" the option, i.e. buy 1 unit of underlying for price $K$, or not to exercise, in which case the option becomes worthless.

A *put* option is similar to a call option; it gives the holder the right to *sell* the underlying. An *American* option is similar to a European option; the holder can exercise at any time up to expiry. Although the techniques developed in Chapter 2 can be applied to American options, I will deal only with European options in this thesis and all references to options should be taken to mean European options.

The payoff to the holder at time $T$ is the net cash flow to the holder.

Suppose the underlying asset has a price process $S_t$, so that its price at time $T$ is $S_T$. Then, for the call option, a decision to exercise gives the holder:

$$S_T - K$$

since he/she pays $K$ for 1 unit of underlying and sells for the current market price $S_T$.

However, if $S_T - K < 0$, it is not beneficial to exercise the option. In this case the payoff is 0.

So the call option payoff can be written as

$$(S_T - K)^+.$$

Similarly, the put option has payoff

$$(K - S_T)^+.$$

A pricing formula for a simple option was developed by Black and Scholes (1973) based on the assumption that the share price follows log-Brownian motion. If $S_t$ is the share price process it satisfies

$$\mathrm{d}S_t = S_t(\sigma \mathrm{d}W_t + \mu \mathrm{d}t)$$

or

$$S_t = S_0 \exp\left[\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right) t\right] \tag{1.1}$$

where $W_t$ is standard Brownian motion and $S_0$, $\mu$ and $\sigma$ are constants. $S_0$ is the initial share price, $\mu$ is the expected rate of return on the share and $\sigma$ is the volatility of the share.

Due to the work by Cox and Ross (1976), Harrison and Kreps (1979) and Harrison and Pliska (1981), we know that under no-arbitrage[2] assumptions there exists a probability measure $\mathbb{P}^*$ (equivalent to the original measure $\mathbb{P}$) under which the discounted payoff of an option is a martingale. The price of a derivative at time $t$ is then given by

$$\mathbb{E}_t^* \left[\mathrm{e}^{-r(T-t)}.\text{payoff}\right] \tag{1.2}$$

where $r$ is the constant riskless rate of interest and the subscript $t$ on the expectation represents that this is the expected value given the information available at time $t$.

Now using the Cameron-Martin-Girsanov Theorem[3], if $\mathbb{P}^*$ is defined by

$$\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}} = Z_T,$$

where

$$Z_t = \exp\left[-\left(\frac{\mu - r}{\sigma}\right) W_t - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 t\right]$$

then $W_t^*$ defined by

[2] Arbitrage is the existence of a riskless trading strategy under which sure profits will be made. The no-arbitrage assumption ensures that if two trading strategies give the same payoff at time $T$ in the future, they must cost the same amount to implement.

[3] See for example, Rogers and Williams Volume 2, Section IV.38.5.

$$W_t = W_t^* - \frac{\mu - r}{\sigma} t$$

is a Brownian motion under $\mathbb{P}^*$ and so

$$S_t = S_0 \exp\left[\sigma W_t^* + \left(r - \frac{1}{2}\sigma^2\right) t\right]. \tag{1.3}$$

The price of a call option is now given by

$$P = \mathbb{E}_t^* \left[\mathrm{e}^{-r(T-t)}(S_T - K)^+\right] \tag{1.4}$$

which can be shown to give the Black-Scholes formula

$$P = S_t \Phi(d_1) - K\mathrm{e}^{-r(T-t)}\Phi(d_2) \tag{1.5}$$

where

$$\begin{aligned} d_1 &= \frac{\ln(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}; \\ d_2 &= d_1 - \sigma\sqrt{T-t} \end{aligned}$$

and $\Phi$ is the cumulative normal distribution function.

## 1.2 Barrier Options

A barrier option is an option whose payoff is either "knocked in" or "knocked out" if the underlying asset price crosses a barrier. This chance that the option may be made worthless reduces the price of the option therefore making it more attractive to some investors. The simplest type has a single constant barrier. For example, an up-and-out call option with constant barrier at $b > S_0$ has a payoff **only** if the underlying asset price remains below $b$ until at least time $T$:

$$\begin{cases} (S_T - K)^+ & \text{if } S_t < b \ \ \forall t \\ 0 & \text{otherwise} \end{cases}$$

An up-and-in has a payoff only if $S_t$ goes above $b$ at some time before $T$:

$$\begin{cases} 0 & \text{if } S_t < b \ \ \forall t \\ (S_T - K)^+ & \text{otherwise} \end{cases}$$

Similarly, a down-and-out with constant barrier at $a < S_0$ has payoff:

$$\begin{cases} (S_T - K)^+ & \text{if } S_t > a \, \forall t \\ 0 & \text{otherwise} \end{cases}$$

and a down-and-in has payoff:

$$\begin{cases} 0 & \text{if } S_t > a \, \forall t \\ (S_T - K)^+ & \text{otherwise} \end{cases}$$

More complicated types of barrier options include those with two barriers, one above and one below the underlying start price and also those with time dependent barriers. Examples of these are partial barriers, which only take effect for an initial period, or which do not come into effect until some specified date, and also moving barriers which could be linear or curved. There is also the rainbow (or outside) barrier option in which there is a second underlying asset which does the knocking out or in of the payoff. We deal here with all cases except the rainbow barrier option. We also omit the case where a rebate is paid to the holder (by the issuer) if the barrier is hit (in the case of a knock-out) or not hit (for a knock-in). The addition of rebates to the method, however, should be fairly easy to implement.

Analytic pricing formulae are available for many types of barrier option using equation (1.2) above. Since we use these formulae later to check the accuracy of our method, I include those used along with a brief description of the method involved.

## 1.3 Some Analytic Pricing Formulae for Barrier Options

### 1.3.1 Notation

For ease of notation, throughout this section I denote the risk-neutral measure by $\mathbb{E}$ or $\mathbb{P}$ so that under this measure

$$S_t = S_0 \exp\left[\sigma W_t + \left(r - \frac{\sigma^2}{2}\right) t\right] \tag{1.6}$$

where $W_t$ is standard Brownian motion. I also let

$$\begin{aligned} X_t &= \sigma W_t + \left(r - \frac{\sigma^2}{2}\right) t \\ &\equiv \sigma(W_t + \eta t) \end{aligned}$$

where

$$\eta = \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right).$$

For any process $Y_t$ let

$$\bar{Y}_t = \sup\{Y_u : 0 \leq u \leq t\}$$

and

$$\hat{Y}_t = \inf\{Y_u : 0 \leq u \leq t\}.$$

$\mathbb{P}^{0,\eta}$ denotes the measure under which $W_t$ is Brownian motion started at 0 with drift $\eta$. $p_t(x,y)$ is the Brownian transition density (the probability that Brownian motion moves from $x$ to $y$ in time $t$) given by

$$p_t(x,y) = (2\pi t)^{-1/2}\exp\left[-\frac{(x-y)^2}{2t}\right].$$

$\Phi(.)$ is the cumulative normal distribution function.

$K$ is the strike for all the options priced here.

### 1.3.2 Barrier Options with a Single Constant Barrier

We now look at the example of a down-and-in call option with lower barrier $a$ below the strike $K$. Using equation (1.2) the price of the down-and-in call option can be written as

$$\begin{aligned} DAIC &= \mathbb{E}\left[\mathrm{e}^{-rT}(S_0\mathrm{e}^{X_T} - K)^+; \hat{X}_T \leq \ln\frac{a}{S_0}\right] \\ &= \int_\kappa^\infty \mathrm{e}^{-rT}(S_0\mathrm{e}^{\sigma y} - K)\mathbb{P}^{0,\eta}[W_T \in \mathrm{d}y, \hat{W}_T \leq \alpha] \end{aligned} \tag{1.7}$$

where

$$\begin{aligned} \kappa &= \frac{1}{\sigma}\ln\frac{K}{S_0}; \text{ and} \\ \alpha &= \frac{1}{\sigma}\ln\frac{a}{S_0}. \end{aligned}$$

To calculate the density, first note[4] that

$$\mathbb{P}^{0,\eta}[W_T \in \mathrm{d}y, \hat{W}_T \leq \alpha] = \mathrm{e}^{\eta y - \eta^2 T/2}\mathbb{P}[W_T \in \mathrm{d}y, \hat{W}_T \leq \alpha]. \tag{1.8}$$

Secondly, using the reflection principle for Brownian motion it can be shown that for $y \geq 0$ and any $a \leq 0$

$$\mathbb{P}[\hat{W}_t \leq a, W_t \geq a + y] = \mathbb{P}[W_t \leq a - y].$$

**Proof**

Theorem I.13.1 from Rogers and Williams Volume 1, states that if we define a process $\tilde{W}$ by

$$\tilde{W}_t := \begin{cases} W_t & t < H_a \\ 2a - W_t & t \geq H_a \end{cases}$$

where $H_a$ is the first hitting time of $a$ by $W$ defined by

$$H_a = \inf\{t > 0 : W_t = a\} \tag{1.9}$$

then $\tilde{W}$ is a Brownian motion. So now

$$\begin{aligned} \mathbb{P}[\hat{W}_t \leq a, W_t \geq a + y] &= \mathbb{P}[\hat{W}_t \leq a, \tilde{W}_t \geq a + y] \\ &= \mathbb{P}[W_t \leq a - y]. \end{aligned}$$

Now substituting $u = a + y$ gives

$$\mathbb{P}[\hat{W}_t \leq a, W_t \geq u] = \mathbb{P}[W_t \leq 2a - u]$$

[4]See Rogers and Williams, Volume 1, Section I.13 for a proof of this.

and for $u \geq a$, $a \leq 0$

$$\begin{aligned}
\mathbb{P}[\hat{W}_t \leq a, W_t \in \mathrm{d}u] &= \mathbb{P}[W_t \in \mathrm{d}(2a-u)] \\
&= p_t(0, 2a-u).
\end{aligned} \tag{1.10}$$

Combining equations (1.7), (1.8) and (1.10) gives

$$\begin{aligned}
DAIC &= \int_{\kappa}^{\infty} \mathrm{e}^{-rT}(S_0 \mathrm{e}^{\sigma y} - K)\mathrm{e}^{\eta y - \eta^2 T/2} p_T(0, 2\alpha - y)\mathrm{d}y \\
&= \frac{1}{\sqrt{2\pi T}} \int_{\kappa}^{\infty} \mathrm{e}^{-rT}(S_0 \mathrm{e}^{\sigma y} - K)\mathrm{e}^{\eta y - \eta^2 T/2} \mathrm{e}^{-\frac{(2\alpha - y)^2}{2T}} \mathrm{d}y
\end{aligned}$$

which can be expressed in terms of normal distribution functions as

$$DAIC = S_0 \left(\frac{a}{S_0}\right)^{\frac{2r}{\sigma^2}+1} \Phi(d_1) - K\mathrm{e}^{-rT} \left(\frac{a}{S_0}\right)^{\frac{2r}{\sigma^2}-1} \Phi(d_2)$$

where

$$\begin{aligned}
d_1 &= \frac{\ln \frac{a^2}{KS_0}}{\sigma\sqrt{T}} + \sigma\sqrt{T}\left(\frac{r}{\sigma^2} + \frac{1}{2}\right) \\
d_2 &= d_1 - \sigma\sqrt{T}
\end{aligned}$$

The formula for the price of a down-and-out call is given by

$$DAOC = BS - DAIC \tag{1.11}$$

where $BS$ is the Black-Scholes price for the standard European call option given by equation (1.5) with $t = 0$.

An up-and-in call option with barrier at $b > K$ can be priced similarly. Denoting its price by $UAIC$ we have

$$\begin{aligned} UAIC &= \mathbb{E}\left[e^{-rT}(S_0 e^{X_T} - K)^+; \bar{X}_T \geq \ln\frac{b}{S_0}\right] \\ &= \int_\kappa^\infty e^{-rT}(S_0 e^{\sigma y} - K)\mathbb{P}^{0,\eta}[W_T \in dy, \bar{W}_T \geq \beta] \end{aligned}$$

where

$$\beta = \frac{1}{\sigma}\ln\frac{b}{S_0}.$$

Again the density is found using the reflection principle which gives, for $a \geq 0$, $u \leq a$,

$$\mathbb{P}[\bar{W}_t \geq a, W_t \in du] = p_t(0, 2a - u).$$

Finally, the price of the up-and-out call option is obtained from

$$UAOC = BS - UAIC$$

### 1.3.3 A Double Sided Barrier Option

The option is knocked out if the price of the share, $S_t$, goes below a lower barrier at $a$ or above an upper barrier at $b$ at any time up to expiry. The price is given by

$$\mathbb{E}\left[e^{-rT}(S_0 e^{X_T} - K)^+; \hat{X}_T \geq \ln\frac{a}{S_0}, \bar{X}_T \leq \ln\frac{b}{S_0}\right]$$

$$
\begin{aligned}
&= \int_{\kappa\vee\alpha}^{\beta} \mathrm{e}^{-rT}(S_0\mathrm{e}^{\sigma y}-K)\mathbb{P}^{0,\eta}\left[W_T\in \mathrm{d}y, \hat{W}_T\geq\alpha, \bar{W}_T\leq\beta\right] \\
&= \int_{\kappa\vee\alpha}^{\beta} \mathrm{e}^{-rT}(S_0\mathrm{e}^{\sigma y}-K)\mathrm{e}^{\eta y-\frac{1}{2}\eta^2 T}\mathbb{P}[W_T\in \mathrm{d}y, \hat{W}_T\geq\alpha, \bar{W}_T\leq\beta]. \quad (1.12)
\end{aligned}
$$

The density in the last line can be expressed as[5]

$$
\frac{1}{\sqrt{2\pi T}}\sum_{k=-\infty}^{\infty}\left\{\exp\left[-\frac{1}{2T}(y+2k(\beta-\alpha))^2\right]-\exp\left[-\frac{1}{2T}(y-2\beta+2k(\beta-\alpha))^2\right]\right\}\mathrm{d}y. \tag{1.13}
$$

Combining equations (1.12) and (1.13) gives the pricing formula

$$
\begin{aligned}
DB &= \sum_{k=-\infty}^{\infty}\{S_0\mathrm{e}^{-2k(\sigma+\eta)(\beta-\alpha)}[\Phi(d_1)-\Phi(d_2)] \\
&\quad -K\mathrm{e}^{-rT}\mathrm{e}^{-2k\eta(\beta-\alpha)}[\Phi(d_1+\sigma\sqrt{T})-\Phi(d_2+\sigma\sqrt{T})] \\
&\quad -S_0\mathrm{e}^{(\sigma+\eta)(2\beta-2k(\beta-\alpha))}\left[\Phi\left(d_1-\frac{2\beta}{\sqrt{T}}\right)-\Phi\left(d_2-\frac{2\beta}{\sqrt{T}}\right)\right] \\
&\quad +K\mathrm{e}^{-rT}\mathrm{e}^{\eta(2\beta-2k(\beta-\alpha))}\left[\Phi\left(d_1-\frac{2\beta}{\sqrt{T}}+\sigma\sqrt{T}\right)-\Phi\left(d_2-\frac{2\beta}{\sqrt{T}}+\sigma\sqrt{T}\right)\right]\Bigg\}
\end{aligned} \tag{1.14}
$$

where

$$
\begin{aligned}
d_1 &= \frac{\beta-T(\sigma+\eta)+2k(\beta-\alpha)}{\sqrt{T}}; \text{ and} \\
d_2 &= \frac{\kappa\vee\alpha-T(\sigma+\eta)+2k(\beta-\alpha)}{\sqrt{T}}.
\end{aligned}
$$

Numerical tests must then be used to determine how many terms of the sum are required to give an accurate estimate of the price.

[5] For a proof see Appendix, Section 8.1.

### 1.3.4 The Single Moving Barrier

This call option is knocked out if the log price of the share, $X_t$, goes below a linear barrier $\alpha + \beta t$ at any time before expiry. The price is given by

$$\begin{aligned}
& \mathbb{E}\left[e^{-rT}(S_0 e^{X_T} - K)^+; X_t \geq \alpha + \beta t \ \forall t \leq T\right] \\
= & \int_{\kappa \vee \theta}^{\infty} e^{-rT}(S_0 e^{\sigma y} - K)\mathbb{P}\left[\frac{X_T}{\sigma} \in \mathrm{d}y, \frac{X_t}{\sigma} \geq \frac{\alpha}{\sigma} + \frac{\beta}{\sigma} t \ \forall t \leq T\right] \\
= & \int_{\kappa \vee \theta}^{\infty} e^{-rT}(S_0 e^{\sigma y} - K)\mathbb{P}\left[W_T + \eta T \in \mathrm{d}y, W_t + \left(\eta - \frac{\beta}{\sigma}\right) t \geq \frac{\alpha}{\sigma} \forall t \leq T\right] \\
= & \int_{\kappa \vee \theta}^{\infty} e^{-rT}(S_0 e^{\sigma y} - K)\mathbb{P}\left[W_T + \gamma T \in \mathrm{d}\left(y - \frac{\beta}{\sigma} T\right), \inf_{t \leq T} W_t + \gamma t \geq \frac{\alpha}{\sigma}\right] \\
= & \int_{\kappa \vee \theta}^{\infty} e^{-rT}(S_0 e^{\sigma y} - K)\mathbb{P}^{0,\gamma}\left[W_T \in \mathrm{d}\left(y - \frac{\beta}{\sigma} T\right), \hat{W}_T \geq \frac{\alpha}{\sigma}\right]
\end{aligned}$$

where

$$\begin{aligned}
\theta &= \frac{\alpha}{\sigma} + \frac{\beta}{\sigma} T; \\
\gamma &= \eta - \frac{\beta}{\sigma}
\end{aligned}$$

and the probability density involved is calculated similarly to the density for the down-and-in call option price and can be expressed as

$$\left\{p_T\left(0, y - \frac{\beta}{\sigma} T\right) - p_T\left(2\frac{\alpha}{\sigma}, y - \frac{\beta}{\sigma} T\right)\right\} e^{\gamma\left(y - \frac{\beta}{\sigma} T\right) - \frac{\gamma^2 T}{2}}.$$

In terms of the standard normal distribution, the price is now given by

$$
\begin{aligned}
S_0 \quad & \Phi\left[\frac{T(\eta+\sigma)-\kappa\vee\theta}{\sqrt{T}}\right] - S_0 \mathrm{e}^{2\alpha(\gamma+\sigma)/\sigma}\Phi\left[\frac{T(\eta+\sigma)+2\alpha/\sigma-\kappa\vee\theta}{\sqrt{T}}\right] \\
& -K\mathrm{e}^{-rT}\Phi\left[\frac{\eta T-\kappa\vee\theta}{\sqrt{T}}\right] + K\mathrm{e}^{-rT+2\alpha\gamma/\sigma}\Phi\left[\frac{\eta T+2\alpha/\sigma-\kappa\vee\theta}{\sqrt{T}}\right].
\end{aligned}
$$

### 1.3.5 Barrier Option with only a Partial Barrier

For an option whose constant barrier at $a < S_0$ stretches from 0 to some point $T/n$ before expiry, a closed form solution for the price is given by

$$
\begin{aligned}
P &= \mathbb{E}\left[\mathrm{e}^{-rT}(S_0\mathrm{e}^{X_T}-K)^+; S_t \geq a \,\forall\, t \leq \frac{T}{n}\right] \\
&= \int_\kappa^\infty \mathrm{e}^{-rT}(S_0\mathrm{e}^{\sigma z}-K)\int_\alpha^\infty \mathbb{P}^{y,\eta}\left[W_{T-\frac{T}{n}}\in \mathrm{d}z\right]\mathbb{P}^{0,\eta}\left[W_{\frac{T}{n}}\in \mathrm{d}y; \hat{W}_{T/n}\geq\alpha\right].
\end{aligned} \tag{1.15}
$$

The densities in this integral are known to be

$$
\mathbb{P}^{y,\eta}\left[W_{T-\frac{T}{n}}\in \mathrm{d}z\right] = \frac{\mathrm{e}^{-\frac{(z-y)^2}{2(T-T/n)}}}{\sqrt{2\pi(T-T/n)}}\mathrm{e}^{\eta(z-y)-\frac{1}{2}\eta^2(T-\frac{T}{n})} \tag{1.16}
$$

$$
\mathbb{P}^{0,\eta}\left[W_{\frac{T}{n}}\in \mathrm{d}y; \hat{W}_{T/n}\geq\alpha\right] = \{p_{\frac{T}{n}}(0,y)-p_{\frac{T}{n}}(2\alpha,y)\}\mathrm{e}^{\eta y-\frac{1}{2}\eta^2\frac{T}{n}} \tag{1.17}
$$

where the latter is obtained from equation (1.8). Combining equations (1.15), (1.16) and (1.17) we can express the price more simply as

$$
P = \frac{\mathrm{e}^{-rT-\frac{1}{2}\eta^2 T}}{\sqrt{2\pi T}}\int_\kappa^\infty \mathrm{d}z(S_0\mathrm{e}^{\sigma z}-K)\mathrm{e}^{\eta z-\frac{z^2}{2T}}\left[\Phi\left(-d_1\right)-\mathrm{e}^{\frac{2\alpha(z-\alpha)}{T}}\Phi\left(-d_2\right)\right]
$$

where

$$
\begin{aligned}
d_1 &= \frac{n\alpha - z}{\sqrt{T(n-1)}}; \text{ and} \\
d_2 &= \frac{2\alpha - n\alpha - z}{\sqrt{T(n-1)}}
\end{aligned}
$$

## 1.4 The Binomial Tree Method of Option Pricing

It may not always be possible to obtain a price analytically from equation (1.2). The binomial tree method of option pricing, developed by Cox, Ross and Rubinstein (1979), enables us to price derivatives numerically by approximating the underlying share price by a discrete-time process. This method provides the basis for the random time-step adaptation described in Chapter 2.

Firstly, the continuous process $S_t$ in equation (1.3) above is approximated by a random walk. Time to expiry, $T$, is divided into $N$ equal steps of length $\Delta t$ and at each time step, the price is assumed to move from $S$ either up to $Su$ with probability $p$ or down to $Sd$ with probability $1-p$. Figure (1-1) below shows the first two steps of a tree where the initial share price is $S$.

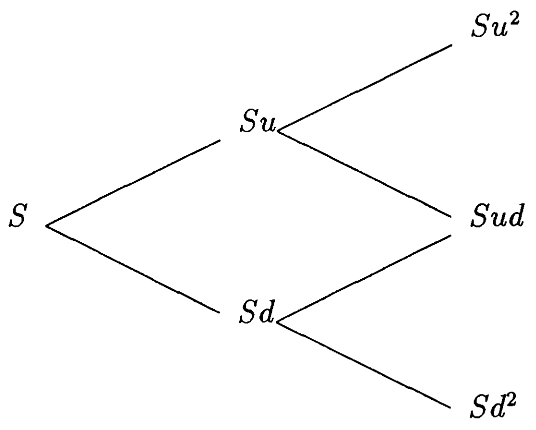


Figure 1-1: A Section of the Binomial Tree

The three parameters $p$, $u$ and $d$ must be chosen so that in the limit as $N \to \infty$

the approximated process has the same mean and variance as $S_t$. Cox, Ross and Rubinstein achieve this by taking

$$\begin{aligned} u &= 1/d = \mathrm{e}^{\sigma\sqrt{\Delta t}}; \text{ and} \\ p &= \frac{1}{2} + \frac{1}{2\sigma}\left(r - \frac{1}{2}\sigma^2\right)\sqrt{\Delta t}. \end{aligned}$$

An alternative way of choosing $p$, $u$ and $d$ is to match the first and second moments of the two processes *over one time step.* If we again assume $u = 1/d$ but now let $u = \mathrm{e}^{\Delta x}$ then this gives us the two equations

$$\begin{aligned} \mathbb{E}^*\left[\mathrm{e}^{\sigma W^*_{\Delta t} + \left(r - \frac{1}{2}\sigma^2\right)\Delta t}\right] &= p\mathrm{e}^{\Delta x} + (1-p)\mathrm{e}^{-\Delta x} \\ \mathbb{E}^*\left[\mathrm{e}^{2\sigma W^*_{\Delta t} + 2\left(r - \frac{1}{2}\sigma^2\right)\Delta t}\right] &= p\mathrm{e}^{2\Delta x} + (1-p)\mathrm{e}^{-2\Delta x}. \end{aligned}$$

The identity $\mathbb{E}^*\left[\mathrm{e}^{\alpha W^*_t}\right] = \mathrm{e}^{\frac{1}{2}\alpha^2 t}$ gives the equations

$$\begin{aligned} \mathrm{e}^{r\Delta t} &= p\mathrm{e}^{\Delta x} + (1-p)\mathrm{e}^{-\Delta x} \\ \mathrm{e}^{(2r+\sigma^2)\Delta t} &= p\mathrm{e}^{2\Delta x} + (1-p)\mathrm{e}^{-2\Delta x} \end{aligned}$$

which are solvable for $p$ and $\Delta x$, given $\Delta t$. Note that we have used the risk-neutral measure. To change from the risk-neutral to the non risk-neutral model, it is only necessary to substitute $\mu$ for $r$ in the above expressions.

Now we define the value function $\phi$ as follows:

$$\begin{aligned} \phi(i,j) &= \mathbb{E}^*[(S_T - K)^+ | S_{i\Delta t} = S_0\mathrm{e}^{j\Delta x}] \\ &= p\phi(i+1, j+1) + (1-p)\phi(i+1, j-1) \qquad (1.18) \end{aligned}$$

for $i = 0, ..., N$ and for $j = -i, ..., i$.

Then at expiry we know that

$$\begin{aligned} \phi(N,j) &= \mathbb{E}^*[(S_{N\Delta t} - K)^+ | S_{N\Delta t} = S_0 e^{j\Delta x}] \\ &= (S_0 e^{j\Delta x} - K)^+ \end{aligned} \tag{1.19}$$

for $j = -N, ..., N$, and the price is given by

$$e^{-rN\Delta t}\phi(0,0) \tag{1.20}$$

which is calculated by working back through the tree.

### 1.4.1 Applying the binomial tree method to barrier options

Applying the above method to pricing barrier options is simple. For example, suppose we want to price an up-and-out call option with constant upper barrier at $b > S_0$. Then if at time step $i$, the share price is given by

$$S_{i\Delta t} = S_0 e^{j\Delta x}$$

which must be below $b$ for the option to remain valid, the equivalent condition on $j$ is

$$j < \frac{1}{\Delta x} \ln \frac{b}{S_0}$$

and the algorithm for the value function $\phi^b$ defined by

$$\phi^b(i,j) = \mathbb{E}^*[(S_T - K)^+ I_{\{\bar{S}_T \le b\}} | S_{i\Delta t} = S_0 e^{j\Delta x}]$$

is given by

$$\phi^b(i,j) = \begin{cases} p\phi(i+1,j+1) + (1-p)\phi(i+1,j-1) & \text{if } j \le \frac{1}{\Delta x}\ln\frac{b}{S_0} \\ 0 & \text{otherwise} \end{cases} \tag{1.21}$$

and

$$\phi^b(N,j) = \begin{cases} (S_0 e^{j\Delta x} - K)^+ & \text{if } j \le \frac{1}{\Delta x}\ln\frac{b}{S_0} \\ 0 & \text{otherwise} \end{cases} . \tag{1.22}$$

However, applying the binomial tree in this way can lead to a bias in the price obtained, as pointed out by Boyle and Lau (1994). The problem occurs due to inaccuracies in calculating the probability of the barrier being crossed and the option knocked out. The bias is greatest if (in the case of the example of the up-and-out call) the barrier falls just higher than a level of lattice points (position A in Figure 1-2). If this happens, then the probability of the share price moving from position A to position B without crossing the barrier is very small. However, with the algorithm $\phi^b$, we are assuming that a movement from position $A$ to position $B$ does indicate that the barrier has not been crossed. Therefore, we are estimating the probability of crossing the barrier to be the same as the probability of an up movement in the tree, whereas in fact the probability of crossing the barrier is much higher than this. This problem is reduced greatly when the upper barrier is positioned just below a level of lattice points, so that the probability of crossing the barrier from a node one level below the barrier is very close to the probability of an up movement in the tree. Boyle and Lau (1994) make use of this fact to develop a method which is significantly more accurate than the standard binomial tree. The random time steps method described in Chapter 2 is more simple to implement than Boyle and Lau's method and also solves this problem but in this case by adjusting the probabilities close to the barriers.

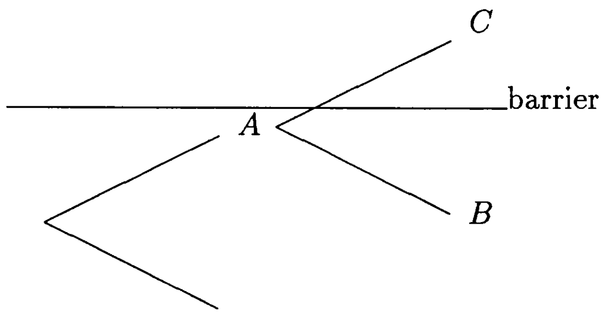


Figure 1-2: The Binomial Tree Close to the Barrier

# Chapter 2

# The Modified Binomial Tree

In the standard binomial tree described in Chapter 1, the time step $\Delta t$ is fixed and $\Delta x$, the price movement, is calculated to ensure a risk-neutral environment. In this alternative approach, $\Delta x$ is fixed and the time steps are random and dependent on the price process $S_t$. This method enables faster, more accurate pricing of, in particular, barrier options. The new binomial tree more accurately describes the situation close to barriers by adjusting the probabilities there.

## 2.1 Set Up

As in Section 1.3, I assume throughout this chapter that $\mathbb{E}$ and $\mathbb{P}$ denote the risk-neutral measure. The price process $S_t$ is defined as in equation (1.6) by

$$S_t = S_0 \exp\left[\sigma W_t + \left(r - \frac{\sigma^2}{2}\right) t\right]$$

and again I denote

$$X_t = \sigma W_t + \left(r - \frac{\sigma^2}{2}\right) t$$

but now let $\mu = r - \sigma^2/2$ be the drift of the process.

Now we fix some $\Delta x > 0$ and look at $X(t)$ only at the points in time at which it has moved by $\pm\Delta x$. Formally, define a process $(\tau_n), n \geq 0$ by

$$\begin{aligned} \tau_0 &= 0 \\ \tau_n &:= \inf\{t > \tau_{n-1} : |X(t) - X(\tau_{n-1})| \geq \Delta x\} \quad n = 1, 2, \cdots \end{aligned} \tag{2.1}$$

and let $(\xi_n)_{n\geq 0}$ be the random walk given by

$$\xi_n = X(\tau_n) \qquad n \geq 0. \tag{2.2}$$

This is the alternative binomial tree with random time steps $(\tau_n - \tau_{n-1})_{n\geq 1}$ and up and down movements $\pm\Delta x$.

The probabilities of these movements are calculated using the scale function $s$. Let $s$ be an increasing function such that $s(X_t)$ is a martingale. Itô's formula gives

$$\begin{aligned} \mathrm{d}s(X_t) &= s'(X_t)\{\sigma \mathrm{d}W_t + \mu \mathrm{d}t\} + \frac{1}{2}s''(X_t)\sigma^2\mathrm{d}t \\ &\equiv \sigma s'(X_t)\mathrm{d}W_t \end{aligned}$$

since for $s(X_t)$ to be a martingale $\mathrm{d}s(X_t)$ must not contain a coefficient of $\mathrm{d}t$, only of $\mathrm{d}W_t$. Now $s$ must satisfy the differential equation

$$\mu s' + \frac{1}{2}\sigma^2 s'' \equiv 0 \tag{2.3}$$

which has solutions

$$s(x) = \begin{cases} A + B\exp\left[-\frac{2\mu x}{\sigma^2}\right] & \text{for } \mu \neq 0 \\ \\ Ax + B & \text{otherwise.} \end{cases} \tag{2.4}$$

So for $\mu \neq 0$ we can take

$$s(x) = -\exp\left[-\frac{2\mu x}{\sigma^2}\right] \tag{2.5}$$

which we will write as $s(x) = -\mathrm{e}^{-2cx}$.

Next, by the Optimal Stopping Theorem, if $p$ is the probability of our process $X_t$ hitting some point $b$ before it hits $-a$, having started from 0, then

$$\begin{aligned} \mathbb{E}s(X_0) = s(0) &= \mathbb{E}s(X_{H_b \wedge H_{-a}}) \\ &= ps(b) + (1-p)s(-a) \end{aligned}$$

where $H_a$ represents the hitting time of $a$, defined by

$$H_a = \inf\{t > 0 : X_t = a\}.$$

This gives

$$p = \frac{s(0) - s(-a)}{s(b) - s(-a)} \tag{2.6}$$

and therefore the probability of an up movement is

$$p = \frac{s(0) - s(-\Delta x)}{s(\Delta x) - s(-\Delta x)} = \frac{\mathrm{e}^{2c\Delta x} - 1}{\mathrm{e}^{2c\Delta x} - \mathrm{e}^{-2c\Delta x}}. \tag{2.7}$$

The key advantage of this binomial lattice over the standard binomial model, is that we can calculate exactly the probability of hitting the barrier. If, as in Figure 2-1, we are at A, a distance $\xi$ (where $\xi < \Delta x$) below the barrier, the probability of hitting the barrier before hitting the lattice point B is

$$p' = \frac{s(0) - s(-\Delta x)}{s(\xi) - s(-\Delta x)}. \tag{2.8}$$

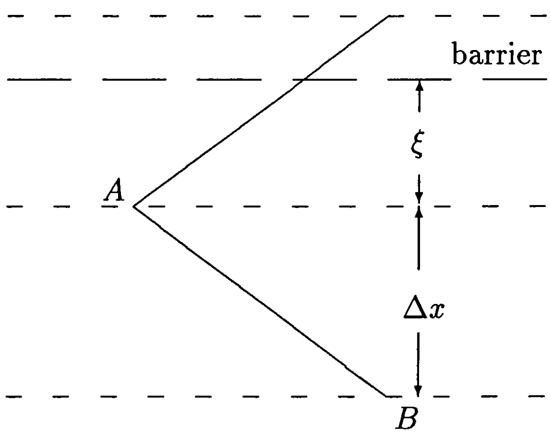


Figure 2-1: The Binomial Tree Close to the Barrier

Before we can use the new binomial tree, we must show that the random time steps $(\tau_n - \tau_{n-1})_{n\geq 1}$ are independent and identically distributed. Proposition 2.1.2 proves this, but first we need the following result:

**Lemma 2.1.1** *Let*

$$\psi_\lambda(x) = \mathbb{E}^x[\mathrm{e}^{-\lambda\tau}] \tag{2.9}$$

*where* $\mathbb{E}^x$ *denotes starting with our process* $X$ *at* $x \in [-\Delta x, \Delta x]$ *at some time* $u$ *and* $\tau$ *is defined similarly to equation (2.1) by*

$$\tau := \inf\{s > u : |X_s| = \Delta x\}.$$

*Then the process $M_t$ defined by*

$$M_t = \mathrm{e}^{-\lambda(t\wedge\tau)}\psi_\lambda(X_{t\wedge\tau}) \qquad (2.10)$$

*is a martingale.*

**Proof:**

$$\mathbb{E}[\mathrm{e}^{-\lambda\tau}|\mathcal{F}_t] = I_{\{\tau\leq t\}}\mathrm{e}^{-\lambda\tau} + I_{\{\tau>t\}}\mathrm{e}^{-\lambda t}\psi_\lambda(X_t) \equiv M_t. \qquad (2.11)$$

□

**Proposition 2.1.2** *The random variables $(\tau_{n+1} - \tau_n)_{n\geq 0}$ are independent and identically distributed, with distribution given by*

$$\mathbb{E}[\mathrm{e}^{-\lambda\tau}] = \frac{\cosh(\mu\sigma^{-2}\Delta x)}{\cosh(\gamma\Delta x)} = \frac{\cosh(c\Delta x)}{\cosh(\gamma\Delta x)} \qquad (2.12)$$

*where $\gamma = \sqrt{\mu^2 + 2\lambda\sigma^2}/\sigma^2$. The common mean is*

$$\mathbb{E}[\tau] \equiv \frac{\Delta x}{\mu}\mathrm{tanhc}(\Delta x) \qquad (2.13)$$

*and common second moment is*

$$\mathbb{E}[\tau^2] = 2(\mathbb{E}[\tau])^2 + \frac{\sigma^2\Delta x}{\mu^3}\tanh(c\Delta x) - \left(\frac{\Delta x}{\mu}\right)^2. \qquad (2.14)$$

*Moreover, they are independent of the random walk $\xi$.*[1]

**Proof:** Let $\tau$, $\psi_\lambda$ and $M_t$ all be defined as above. Applying Itô's formula to the martingale $M_t$ gives

[1]This is the one-dimensional case of Reuter's Theorem. For details, and the multi-dimensional proof, see Rogers and Williams Volume 2, page 84.

$$\begin{aligned}
\mathrm{d}M_t &= -\lambda \mathrm{e}^{-\lambda t}\psi_\lambda(X_t)\mathrm{d}t + \mathrm{e}^{-\lambda t}\psi'_\lambda(X_t)\{\sigma \mathrm{d}W_t + \mu \mathrm{d}t\} \\
&\qquad\qquad\qquad\qquad + \frac{1}{2}\mathrm{e}^{-\lambda t}\psi''_\lambda(X_t)\{\sigma^2 \mathrm{d}t\} \\
&= \sigma \mathrm{e}^{-\lambda t}\psi'_\lambda(X_t)\mathrm{d}W_t.
\end{aligned}$$

Therefore $\psi_\lambda$ must satisfy the differential equation

$$0 = -\lambda\psi_\lambda + \mu\psi'_\lambda + \frac{1}{2}\sigma^2\psi''_\lambda \equiv \mathcal{G}\psi_\lambda - \lambda\psi_\lambda \tag{2.15}$$

which has general solution

$$\psi_\lambda(x) = A\mathrm{e}^{(\gamma-c)x} + B\mathrm{e}^{(-\gamma-c)x}$$

where

$$\begin{aligned}
\gamma &= \frac{\sqrt{\mu^2 + 2\lambda\sigma^2}}{\sigma^2} \\
c &= \frac{\mu}{\sigma^2}.
\end{aligned}$$

Now $\mathbb{E}[\mathrm{e}^{-\lambda\tau}] = \psi_\lambda(0)$ where $\psi_\lambda$ is the solution to equation (2.15) satisfying the boundary conditions

$$\psi_\lambda(\Delta x) = \psi_\lambda(-\Delta x) = 1. \tag{2.16}$$

This gives us

$$\mathbb{E}[\mathrm{e}^{-\lambda\tau}] = \frac{\cosh(c\Delta x)}{\cosh(\gamma\Delta x)} \equiv f(\lambda) \tag{2.17}$$

and then the first and second moments of $\tau$ are given by

$$\begin{aligned} \mathrm{E}[\tau] &= -f'(0) \\ \mathrm{E}[\tau^2] &= f''(0) \end{aligned}$$

which when calculated give equations (2.13) and (2.14).

Finally, we need to show that the random variables $(\tau_{n+1} - \tau_n)_{n\geq 0}$ are independent of the random walk $\xi_n$. This is equivalent to showing that the distribution of the time step $\tau_1$ is the same whether the first step is an up or a down movement.

First, we calculate

$$\mathbb{E}[\mathrm{e}^{-\lambda\tau_1} : X(\tau_1) = -\Delta x]$$

by solving the differential equation (2.15) with boundary conditions

$$\psi_\lambda(\Delta x) = 0; \quad \psi_\lambda(-\Delta x) = 1$$

since if we start at $\Delta x$, $X(\tau_1) = \Delta x$ and if we start at $-\Delta x$, $X(\tau_1) = -\Delta x$ and $\tau_1 = 0$.

The solution is

$$\mathbb{E}[\mathrm{e}^{-\lambda\tau_1} : X(\tau_1) = -\Delta x] = \frac{1}{2\mathrm{e}^{c\Delta x}\cosh(\gamma\Delta x)}.$$

Now

$$\mathbb{E}[\mathrm{e}^{-\lambda\tau_1} | X(\tau_1) = -\Delta x] = \frac{\mathbb{E}[\mathrm{e}^{-\lambda\tau_1} : X(\tau_1) = -\Delta x]}{\mathbb{P}[X(\tau_1) = -\Delta x]}$$

$$
\begin{aligned}
&= \left[\frac{1}{2e^{c\Delta x}\cosh(\gamma\Delta x)}\right] \Big/ \left[\frac{1-e^{-2c\Delta x}}{e^{2c\Delta x}-e^{-2c\Delta x}}\right] \\
&= \frac{\cosh(c\Delta x)}{\cosh(\gamma\Delta x)}
\end{aligned}
$$

using equation (2.7) for the second line. Since replacing $c$ with $-c$ simply gives us minus the drift, we have

$$
\begin{aligned}
\mathbb{E}[e^{-\lambda\tau_1}|X(\tau_1)=+\Delta x] &= \frac{\cosh(-c\Delta x)}{\cosh(\gamma\Delta x)} \\
&= \frac{\cosh(c\Delta x)}{\cosh(\gamma\Delta x)}.
\end{aligned}
$$

Therefore, the law of $\tau_1$ is independent of $X(\tau_1) \equiv \xi_1$.

□

## 2.2 Application of the alternative binomial tree

Now it is possible to price, for example, an up-and-out call option with initial share price $S$, strike price $K$, time to expiry $T$ and barrier at $H > 0$.

The price of such an option is given by

$$
\mathbb{E}\left[e^{-rT}(Se^{X_T}-K)^+; \bar{X}_T \leq \ln\frac{H}{S}\right] \tag{2.18}
$$

and so, if we assume there have been $N_T$ random time steps $\tau$ before expiry, using the random walk we constructed we can write down

$$
\phi(0,j) = (Se^{j\Delta x}-K)^+ \mathrm{I}_{\{j\Delta x < \ln\frac{H}{S}\}} \tag{2.19}
$$

for $j \in \mathbb{Z}$, where $\phi(i,j)$ is the expected payoff from the option if the random walk has a value $j\Delta x$ with $i$ time steps left before expiry, specifically, $\phi(0,j)$ is the actual payoff if the random walk has value $j\Delta x$ at expiry. Note that since the barrier is absorbing we can assume that if the random walk is below the barrier at time $T$, it has not crossed it anywhere in its path.

To find the earlier values of the payoff process the following algorithm is used

$$\phi(i+1,j) = p.\phi(i,j+1) + (1-p).\phi(i,j-1) \tag{2.20}$$

unless $\ln\frac{H}{S} - j\Delta x < \Delta x$ in which case

$$\phi(i+1,j) = (1-p').\phi(i,j-1) \tag{2.21}$$

where $p'$ is given by equation (2.8) with $\xi = \ln\frac{H}{S} - j\Delta x$, since from this position the payoff will be 0 unless the next movement is down without touching the barrier.

It now follows from Proposition 1 that the price of the barrier option can approximated by

$$\sum_{n\geq 0} \mathbb{P}(N_T = n)\,\phi(n,0). \tag{2.22}$$

The dynamic programming recursion given by equations (2.19) to (2.21) allows us to compute $\phi$, so we only need to calculate $\mathbb{P}(N_T = n)$ for all $n$. It amounts to the same thing to compute $\mathbb{P}(N_T \geq n) = \mathbb{P}(\tau_n < T)$, and this is made much easier by the fact that the increments of the sequence $(\tau_n)$ are independent with the same law, characterised by (2.12). If we abbreviate $\mu_\tau \equiv E[\tau_1]$ and $\sigma_\tau^2 \equiv \text{var }(\tau_1)$, then by the Central Limit Theorem we shall have that approximately

$$\frac{(\tau_n - n\mu_\tau)}{\sigma_\tau\sqrt{n}} \sim N(0,1).$$

However, this approximation turns out to be rather too crude, and a refinement of the Central Limit Theorem is required. Fortunately, such refinements are well developed; Petrov (1995), Chapter 5, gives a good account of expansions of the Central Limit Theorem. In particular, Theorem 5.21 of Petrov states the following

$$\mathbb{P}\left(\frac{(\tau_n - n\mu_\tau)}{\sigma_\tau\sqrt{n}} \leq x\right) = \Phi(x) + \frac{\alpha_3(1-x^2)e^{-x^2/2}}{\sqrt{72\pi n}} + o(n^{-1/2}),$$

where

$$\alpha_3 \equiv \mathbb{E}\left[\left(\frac{\tau_1 - \mu_\tau}{\sigma_\tau}\right)^3\right]$$

is the third moment of the centred and scaled random time-step. Higher order terms in the expansion are available, but we found that they made no appreciable difference to the accuracy of the results, so have omitted them entirely. Clearly, we can compute the value of $\alpha_3$ explicitly from (2.12); firstly, multiplying out the expectation gives us

$$\alpha_3 = \frac{\mathbb{E}[\tau_1^3] - 3\mu_\tau\mathbb{E}[\tau_1^2] + 2\mu_\tau^3}{\sigma_\tau^3}$$

and using Maple to differentiate with respect to $\lambda$ we obtain

$$\mathbb{E}[\tau_1^3] = -5\frac{\Delta x^2}{\mu^2}\mu_\tau + 6\mu_\tau^3 - 3\frac{\Delta x^2\sigma^2}{\mu^4} + \frac{6\sigma^2}{\mu^2}\mu_\tau^2 + \frac{3\sigma^4}{\mu^4}\mu_\tau.$$

The results of this method applied to several options are given in the next chapter. For comparison with other methods it is convenient to have some idea of the number of time steps being used. Suppose we wish to have approximately $N$ time steps. So we want $\mathbb{E}[\tau_1]$ to be approximately equal to $T/N$. From equation (2.13), we require

$$\begin{aligned}\frac{T}{N} \simeq \frac{\Delta x}{\mu}\tanh(c\Delta x) &\simeq \frac{c\Delta x^2}{\mu} \\ &= \frac{\Delta x^2}{\sigma^2}\end{aligned}$$

and so we choose $\Delta x = \sigma\sqrt{T/N}$.

## 2.3 Extension to more complex barriers

The new binomial tree can be applied equally well to the more complex barrier options defined in Chapter 1. The most simple extension is to the double barrier option with constant barriers. For this the pricing algorithm is identical to that given by equations (2.19) to (2.21) except that, with lower barrier at $H'$, if $j\Delta x - \ln\frac{H'}{S} < \Delta x$, i.e. if we are within one lattice step of the barrier

$$\phi(i+1,j) = p''.\phi(i,j+1)$$

where $p''$ is given by

$$\frac{s(0)-s(-\xi)}{s(\Delta x)-s(-\xi)}$$

with $\xi = j\Delta x - \ln\frac{H'}{S}$.

For any time-dependent barrier the random time steps in this model cause a slight problem in that we do not know at any given lattice point how much time has elapsed and therefore where the barrier is. To deal with this, we simply assume that at time step $i$, $i\mathbb{E}[\tau]$ time has elapsed. Therefore, for the partial barrier option whose constant barrier stretches, say, from time 0 to time $T/2$, we need only check at each lattice point whether

$$i\mathbb{E}[\tau] < \frac{T}{2}$$

in which case the barrier is still effective. For a linear barrier with equation $\alpha+\beta t$, at time step $i$ the barrier is assumed to be at a level

$$\alpha + \beta i\mathbb{E}[\tau].$$

# Chapter 3

# Results and Conclusions

## 3.1 Results

The computations reported here were performed on a Sun Sparcserver 1000E. We started by computing the values of a European call option with the following parameters:

Table 3.1: Parameter values

| | |
|---|---|
| $S$ | 95 |
| $K$ | 100 |
| $\sigma$ | 0.25 per year |
| $T$ | 1 year |
| $r$ | 0.1 per year |
| $n$ | number of time steps* |

* For the modified binomial method, this figure is the estimated average number of time steps, since the number of random time steps there will be is not known exactly.

The modified binomial method is not a significant improvement over the standard binomial method for the pricing of the standard call as the modification is tailored specifically to increase accuracy when barriers are involved. However, we can see from Table 3.2 that the modified binomial method does give accurate results for the standard call option. The figures in brackets below the prices are the times in CPU time taken for the calculations. The Black-Scholes price is given in the following table along with the price obtained directly from the integral solution using numerical integration.

Table 3.2: Standard European Call

| $n$ | Standard Binomial | Modified Binomial |
|---|---|---|
| 25 | 11.5462 (0.0016) | 11.5992 (0.0034) |
| 50 | 11.6681 (0.0020) | 11.6496 (0.0063) |
| 75 | 11.6743 (0.0028) | 11.6480 (0.0106) |
| 100 | 11.6317 (0.0039) | 11.6443 (0.0162) |
| 150 | 11.6602 (0.0069) | 11.6549 (0.0322) |
| 200 | 11.6646 (0.0111) | 11.6519 (0.0535) |
| 400 | 11.6520 (0.0404) | 11.6546 (0.2022) |
| 800 | 11.6556 (0.1655) | 11.6564 (0.7714) |
| 1600 | 11.6565 (0.6949) | 11.6569 (8.9725)* |
| 3200 | 11.6573 (6.3333)* | 11.6572 (11.0)* |

Table 3.3: Accurate values

| Method | Price | Time taken |
|---|---|---|
| Black-Scholes | 11.65735 | 0.001 |
| Numerical Integration | 11.65737 | 0.001 |

* The CPU times for these last three calculations seem to be too large relative to the previous entries in the table. This could be due to the machines used putting a lower priority on tasks which take more than a certain length of time. We would expect the time taken to be multiplied by 4 when the number of steps used is doubled.

Next we price the down and out call option, with the same parameter values as the standard call and with lower barrier set at 90. Both the down and out call option and the standard call priced are identical to those priced in Boyle and Lau (1994). In the table below we compare the standard and modified binomial methods. Also included are Boyle and Lau's results and the results from Derman et al's method of interpolation. As a final comparison in the final column of Table 3.4 we show the results obtained from using the standard binomial tree but with altered probabilities close to the barrier as in the modified binomial method. The only difference between these two methods is that random time steps are used in the modified binomial.

Table 3.4: Down-and-out Call Option

| $n$ | Standard Binomial | Modified Binomial | Boyle and Lau | Derman et al | Altered Binomial |
|---|---|---|---|---|---|
| 25 | 8.8406 (0.0019) | 5.9810 (0.0031) | 6.042 (21 steps) | 7.5513 (0.002) | 5.9890 (0.002) |
| 50 | 7.2372 (0.0020) | 5.9852 (0.0046) | | 6.4936 (0.004) | 5.9951 (0.002) |
| 75 | 6.2981 (0.0026) | 5.9921 (0.0073) | 6.020 (85 steps) | 5.9762 (0.005) | 6.0022 (0.003) |
| 100 | 7.5028 (0.0038) | 5.9932 (0.0100) | | 5.7325 (0.01) | 5.9952 (0.004) |
| 150 | 6.5601 (0.0063) | 5.9945 (0.0185) | 6.006 (192 steps) | 5.8509 (0.02) | 5.9982 (0.007) |
| 200 | 7.2299 (0.0100) | 5.9957 (0.0298) | 5.998 (342 steps) | 5.8709 (0.03) | 5.9999 (0.01) |
| 400 | 6.6501 (0.0355) | 5.9960 (0.1047) | 6.000 (534 steps) | 5.9279 (0.11) | 5.9966 (0.04) |
| 800 | 6.6038 (0.1479) | 5.9966 (0.3963) | | 5.9631 (0.47) | 5.9970 (0.14) |
| 1600 | 6.1723 (0.5778) | 5.9967 (1.5377) | | 5.9855 (2.7) | 5.9969 (0.58) |
| 3200 | 6.2670 (2.3417) | 5.9968 (6.1414) | | * | 5.9969 (2.39) |

Table 3.5: Accurate values

| Method | Price | Time taken |
|---|---|---|
| Black-Scholes type formula | 5.99684 | 0.001 |
| Numerical Integration | 5.99684 | 0.001 |

* I have omitted Derman et al's method using 3200 steps for computational ease since the method uses too much computer memory when many steps are used.

It is immediately obvious that even with only 25 time steps, the modified binomial is much closer to the true price. Although the modified method is slower than the standard method, the fast convergence to the true price makes it the more economical method to use. For example, after 3200 steps of the standard method, and just over 2 seconds of cpu time, the estimated price is 6.2670, still inaccurate by 0.27, nearly 5%; whereas with the modified method, we have greater accuracy after 25 steps, or just 0.03 seconds, and when 1600 steps are used (taking under 2 seconds), we are within 0.002% of the true price. The modified method also compares well to Boyle and Lau's results, having greater accuracy after 200 steps than their method gives with 342 steps. Derman et al's method is also not as accurate as the modified binomial, although significantly better than the standard binomial and just as fast. Finally, the standard binomial method with altered probabilities close to the barrier (final column) is just as accurate as the modified binomial method, and a significant improvement on the standard binomial tree. The advantages of the modified method seem to result, therefore, entirely from the altered probabilities close to the barrier, with no advantage being gained from using random time steps.

Next we tested the method on knock-out call options with two barriers. We compare our results with those of Geman and Yor (1996), scaling the start price, strike and barriers up by 50. The parameters of the three options priced are:

Table 3.6: Parameter values for the double barrier options

| | | | |
|---|---|---|---|
| $\sigma$ | 0.5 | 0.5 | 0.2 |
| $r$ | 0.05 | 0.05 | 0.02 |
| $T$ | 1.0 | 1.0 | 1.0 |
| $S_0$ | 100.0 | 100.0 | 100.0 |
| $K$ | 100.0 | 87.5 | 100.0 |
| Lower barrier | 75.0 | 50.0 | 75.0 |
| Upper barrier | 150.0 | 150.0 | 125.0 |

Again, the results gained compare favourably with Geman and Yor's results and with the accurate price of a double barrier option calculated from the formula derived in Section 1.3.3 of Chapter 1. The modified binomial tree again gives greater accuracy than the standard binomial, and converges more quickly. The times taken for the double barrier option are much quicker than those for the single barrier option, because with two barriers in place the whole tree does not need to be worked through, only that part between the two barriers. This means that as $n$ increases, the time taken is increasing only by order $n$ and not by order $n^2$ as with the single barrier option.

Table 3.7: Double barrier option (a)

| $n$ | Standard Binomial | Modified Binomial |
|---|---|---|
| 25 | 1.9680 (0.0015) | 1.0840 (0.0027) |
| 50 | 1.2619 (0.0017) | 0.9254 (0.0033) |
| 75 | 1.5841 (0.0018) | 0.9513 (0.0042) |
| 100 | 1.4669 (0.0021) | 0.9365 (0.0048) |
| 150 | 1.0634 (0.0027) | 0.8921 (0.0068) |
| 200 | 1.2225 (0.0032) | 0.9089 (0.0087) |
| 400 | 1.1671 (0.0062) | 0.9030 (0.0176) |
| 800 | 0.9673 (0.0146) | 0.8927 (0.0373) |
| 1600 | 1.0291 (0.0382) | 0.8949 (0.0862) |
| 3200 | 0.9224 (0.1026) | 0.8930 (0.2298) |

Table 3.8: Accurate value

| Method | Price | Time taken |
|---|---|---|
| Numerical Integration | 0.8929 | 0.009 |
| Geman and Yor | 0.89 | – |

Table 3.9: Double barrier option (b)

| $n$ | Standard Binomial | Modified Binomial |
|---|---|---|
| 25 | 5.8593<br>(0.0015) | 4.2780<br>(0.0028) |
| 50 | 4.0096<br>(0.0017) | 3.8872<br>(0.0036) |
| 75 | 5.3711<br>(0.0021) | 3.9606<br>(0.0046) |
| 100 | 4.9946<br>(0.0023) | 3.9208<br>(0.0053) |
| 150 | 3.8190<br>(0.0031) | 3.8133<br>(0.0074) |
| 200 | 4.2229<br>(0.0039) | 3.8502<br>(0.0098) |
| 400 | 4.3076<br>(0.0087) | 3.8354<br>(0.0204) |
| 800 | 3.8491<br>(0.0217) | 3.8098<br>(0.0464) |
| 1600 | 3.9877<br>(0.0579) | 3.8142<br>(0.1118) |
| 3200 | 3.8373<br>(0.1593) | 3.8090<br>(0.3027) |

Table 3.10: Accurate value

| Method | Price | Time taken |
|---|---|---|
| Numerical Integration | 3.8086 | 0.0087 |
| Geman and Yor | 3.8075 | – |

Table 3.11: Double barrier option (c)

| $n$ | Standard Binomial | Modified Binomial |
|---|---|---|
| 25 | 2.6102 (0.0016) | 2.1472 (0.0024) |
| 50 | 2.0006 (0.0019) | 2.0518 (0.0032) |
| 75 | 2.2934 (0.0021) | 2.0788 (0.0037) |
| 100 | 2.4289 (0.0025) | 2.0984 (0.0056) |
| 150 | 2.1504 (0.0033) | 2.0670 (0.0059) |
| 200 | 2.1024 (0.0043) | 2.0591 (0.0079) |
| 400 | 2.2292 (0.0097) | 2.0642 (0.0163) |
| 800 | 2.1265 (0.0246) | 2.0578 (0.0390) |
| 1600 | 2.1019 (0.0666) | 2.0558 (0.1012) |
| 3200 | 2.1330 (0.1850) | 2.0558 (0.2871) |

Table 3.12: Accurate value

| Method | Price | Time taken |
|---|---|---|
| Numerical Integration | 2.0544 | 0.0087 |
| Geman and Yor | 2.055 | – |

Finally, we use the new method on some more complicated barrier options. We calculated the prices of a down and out call option with moving linear barrier; a double barrier option with two moving linear barriers; and a partial barrier option where the barrier is only effective for the first half of the time to expiry. The parameter values are the same as for the standard European call. The barrier for the down and out call is given by

$$\ln\frac{90}{95} + 0.1t$$

and the barriers of the double barrier option are given by

$$\ln\tfrac{160}{95} + 0.1t$$

$$\ln\tfrac{90}{95} - 0.1t.$$

In both cases, this is the barrier with respect to $\ln(S_t/S_0)$. The barrier for the partial barrier option is at 90 with respect to $S_t$ for time from 0 to 0.5, and thereafter is not effective.

Table 3.13: Different types of barriers

| | Single moving | Double moving | Partial |
|---|---|---|---|
| 25 | 4.9959<br>(0.002) | 5.2441<br>(0.001) | 6.1404<br>(0.002) |
| 50 | 4.9511<br>(0.002) | 5.3103<br>(0.002) | 6.1202<br>(0.002) |
| 75 | 4.9482<br>(0.003) | 5.3339<br>(0.002) | 6.1267<br>(0.003) |
| 100 | 4.9436<br>(0.003) | 5.3268<br>(0.003) | 6.1296<br>(0.003) |
| 150 | 4.9364<br>(0.004) | 5.3538<br>(0.004) | 6.1292<br>(0.005) |
| 200 | 4.9357<br>(0.005) | 5.3598<br>(0.005) | 6.1322<br>(0.006) |
| 400 | 4.9314<br>(0.01) | 5.3602<br>(0.009) | 6.1323<br>(0.01) |
| 800 | 4.9296<br>(0.02) | 5.3660<br>(0.02) | 6.1330<br>(0.02) |
| 1600 | 4.9286<br>(0.03) | 5.3668<br>(0.03) | 6.1329<br>(0.04) |
| 3200 | 4.9281<br>(0.07) | 5.3672<br>(0.07) | 6.1332<br>(0.08) |
| Accurate* | 4.9277<br>(0.001) | 5.3679<br>(0.005) | 6.1332<br>(0.002) |

* The true value of the double moving barrier option was calculated using the formula derived by Kunitomo and Ikeda (1992).

In all three cases we have accuracy to 2 decimal places with 800 steps (and with only 400 steps for two of the cases). Accuracy to 3 decimal places is achieved with 3200 steps. Even with our slightly crude approximation of the time-dependent barrier, as described in Section 2.3, we still achieve a high level of accuracy with the modified binomial method.

## 3.2 Conclusions

I have presented here a modified binomial tree with random time steps. This enables the more accurate pricing of barrier options since the probability of hitting the barrier can be calculated precisely based on current distance from the barrier. The method adapts well to all kinds of barrier options including those with time-dependent barriers. Full results are given for the standard call, simple down and out call, double barrier option, and time-dependent barrier options with single moving barrier, two moving barriers and partial barrier. For the standard call, results compare well with the standard binomial method, and for all other options the results are more accurate and converge more quickly to the true price. We compare our results against those of Boyle and Lau (1994) and Derman et al (1995) for the simple down and out option, and against those of Geman and Yor (1996) for the double barrier option and find our method compares favourably against all three. For the time-dependent barriers we show that our approximation of the level of the barrier at any moment in time is sufficient and good accuracy is still achieved. Finally, the comparison for the down-and-out call between standard binomial; binomial with altered probabilities; and modified binomial methods shows that the advantage of the modified binomial results entirely from the altered probability close to the barrier and that the method can be made faster by removing the random time steps from the tree, but keeping in the altered probabilities close to the barrier.

# Part II

# Effects of a Trading Time-Lag

# Chapter 4

# Introduction

In Part II of this thesis I am concerned with some effects of relaxing one of the basic assumptions of frictionless markets: immediate trading. In reality, there will always be some small delay between the decision to trade and the completion of that trade. This could be due to illiquidity of the assets involved so that the delay occurs while waiting for cash or other assets to become available, but is also certain to occur in any trade even when otherwise perfect markets are assumed, due to the time taken to communicate between the different parties involved.

We calculate the effects of this time-lagged trading by considering a simple utility maximisation problem. Merton (1969) developed a solution to the problem of an investor who wishes to maximise utility from consumption in a continuous-time setting. The discrete-time version was dealt with by Samuelson (1969). We omit consumption decisions from the problem and look instead at an investor aiming to maximise utility of final wealth by choosing how much of his current wealth to invest in a single risky asset.

We look at two distinct problems. Firstly, using exponential utility, we calculate the effect on the fair price of a put option for an investor trading in a time-lagged market. We look at this problem in both a discrete- and a continuous-time setting. The continuous-time "solution" is not a rigorous proof, involving Taylor's series approximations and some assumptions about the order of the effect of the lag. However, it does give us a better idea about the relationship between the lag and its effect on the price of the put. Comparisons between the two sets of results (discrete and continuous-time) are good.

Secondly, we look at an investor with constant relative risk aversion (i.e. a power utility function). The maximisation problem in this case was dealt with fully in Merton (1969) for the no-lag case. Since this utility is more difficult to work with, we look only at the maximisation problem and obtain the effect of a lag on expected final utility. In this case we work only in discrete time. However, we use two different discrete-time methods for comparison. Firstly, we calculate the value function numerically for four sets of parameter values. The optimal portfolio selection cannot be found precisely but is instead estimated by interpolation between discrete points. Secondly, we approximate the value function by the exponential of a quadratic function in the proportion of wealth invested in the risky asset and, assuming that $h$, the lag, is small, ignore terms of order $h^3$ and above in order to obtain an analytic formula for the initial value function as the exponential of a quadratic in $h$.

In the remainder of this chapter, I define the utility functions looked at here and describe the binomial tree used throughout this part of the thesis. In the following chapter I describe the methods used for the exponential utility function and in Chapter 6 the results are given. In Chapter 7 I derive solutions to the problem of the investor with power utility function, and present the results.

## 4.1 Utility Functions

The utility function of an individual is an indicator of the satisfaction he gains from having a certain level of wealth at a certain time. In this thesis, I look only at utility functions which are constant through time. The function is taken to be strictly increasing and strictly concave to reflect the fact that more wealth is better than less wealth and that one extra unit of wealth is worth more to the poor than to the rich.

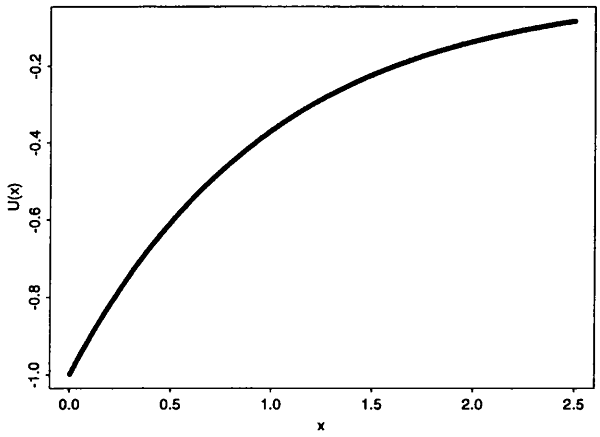


Figure 4-1: Exponential utility function $-e^{-x}$

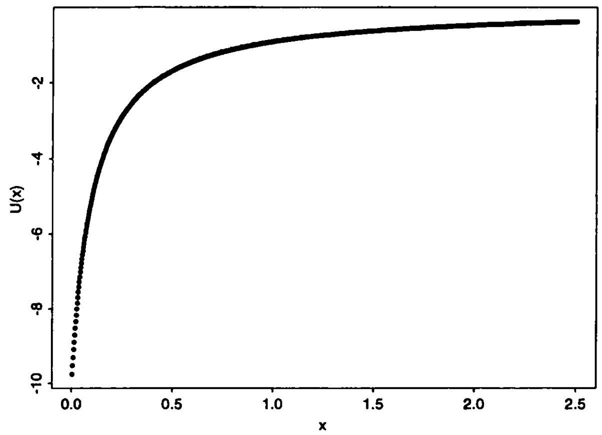


Figure 4-2: Power utility function $-\frac{1}{x}$

## 4.2 The Binomial Tree

In the following Chapters, 5 to 7, the discrete time method used is a binomial tree as described in Chapter 1. In more detail, this is set up as follows.

Let the discrete time step be of length $h$ and the probability of an up movement be $p$. I use the second of the two methods mentioned in Chapter 1. That is, I take an up movement of $a$ and down movement $1/a$ and calculate $p$ and $a$ by matching the first and second moments of the discrete and continuous-time processes. Assume the share price $S_t$ follows exponential Brownian motion given by

$$S_t = S_0 \exp\left[\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right)t\right] \tag{4.1}$$

where $S_0$,$\mu$ and $\sigma$ are constants and $W_t$ is standard Brownian motion. The value of the process $S$ at time $h$, given $S_0 = 1$, has first and second moments

$$\begin{aligned} \mathbb{E}[S_h] &= \exp[\mu h]; \\ \mathbb{E}[S_h^2] &= \exp[2\mu h + \sigma^2 h] \end{aligned}$$

and for the binomial approximation

$$\begin{aligned} \mathbb{E}[S_h] &= pa + \frac{1-p}{a}; \\ \mathbb{E}[S_h^2] &= pa^2 + \frac{1-p}{a^2}. \end{aligned}$$

This gives the simultaneous equations

$$pa + \frac{1-p}{a} = \mathrm{e}^{\mu h}$$

$$pa^2 + \frac{1-p}{a^2} = e^{(2\mu+\sigma^2)h}$$

which are solved by

$$p = \frac{ae^{\mu h} - 1}{a^2 - 1}$$

and a quartic equation in $a$ with the solutions $a = 1$ and -1. Factoring these out we have the quadratic

$$a^2 - a\beta + 1 = 0$$

where $\beta = e^{-\mu h} + e^{(\mu+\sigma^2)h}$.

The roots of this are

$$\begin{aligned} a^+ &= \frac{\beta}{2} + \frac{\sqrt{\beta^2 - 4}}{2} \\ a^- &= \frac{\beta}{2} - \frac{\sqrt{\beta^2 - 4}}{2}. \end{aligned}$$

Since $\beta > 2$ this gives us two positive roots. However, we need $a > 1$ and $p \in (0,1)$ for a usable solution. Now $a^+ > \beta/2 > 1$ and, letting

$$p(a) = \frac{ae^{\mu h} - 1}{a^2 - 1}$$

be the value of the probability $p$ given a value $a$ for the up movement, we have

$$p(e^{\mu h}) = \frac{e^{2\mu h} - 1}{e^{2\mu h} - 1} = 1$$

and since the value of the quadratic in $a$ for $a = e^{\mu h}$ is less than 0 (and because the graph of that quadratic is a U-shape) this means that $a^+ > e^{\mu h}$ and so

$$p(a^+) < p(e^{\mu h}) = 1.$$

Also, $a^+ e^{\mu h} > 1$, $(a^+)^2 > 1$ and so $p(a^+) > 0$. However, $a^- < e^{\mu h}$, giving $p(a^-) > 1$. Together these results show that $a^+$ and $p(a^+)$ are the correct parameters.

Finally, if $r$ is the continuously compounded riskless rate, then the riskless rate over one time step is given by

$$\rho = e^{rh}.$$

# Chapter 5

# Exponential Utility Function Pricing a Put Option

## 5.1 The Problem

In our first example we look at the effect of time-lagged trading on an individual investing in the market. We investigate the fair price of a European put option for such an investor. Suppose this investor sells a put option for some price $p$ at time 0. At time $T$, the expiry date of the option, he must pay out an amount $Y$, where

$$Y = (K - S_T)^+.$$

The investor will aim to maximise the expected utility of his final wealth, $w_T$, by trading in the risky asset. The fair price to the investor is found by comparing his position when he does sell the put option with that when he does not. The price of a very small amount $\varepsilon$ of puts, $p_\varepsilon$, is given by

$$\max \mathbb{E}[U(w_T - \varepsilon Y)|w_0 = x_0 + p_\varepsilon] = \max \mathbb{E}[U(w_T)|w_0 = x_0] \tag{5.1}$$

where maximisation takes place over all possible investment strategies. We call this value $p_\varepsilon$, the *ask* price for $\varepsilon$ puts, i.e. the minimum price a *seller* of the option would require. The *bid* price for $\varepsilon$ puts, $p_\varepsilon^{\mathrm{b}}$ say, is the maximum price a *buyer* would be prepared to pay for $\varepsilon$ put options[1]. It is given by

$$\max\mathbb{E}[U(w_T + \varepsilon Y)|w_0 = x_0 - p_\varepsilon^{\mathrm{b}}] = \max\mathbb{E}[U(w_T)|w_0 = x_0] \qquad (5.2)$$

or, in other words, $p_\varepsilon^{\mathrm{b}} = -p_{-\varepsilon}$.

In a frictionless market with no lag these equations would give us $p_\varepsilon$ equal to $p_\varepsilon^{\mathrm{b}}$ equal to the true price of the put. If we write final wealth $w_T$ as $x_0 + X_T$ where $X_t$ is the gains from trade at time $t$, equation (5.1) becomes

$$\max_{\{X_t\}} \mathbb{E}[U(x_0 + X_T)] = \max_{\{X_t\}} \mathbb{E}[U(x_0 + p_\varepsilon + X_T - \varepsilon Y)]. \qquad (5.3)$$

Since the payoff of a put option can be replicated by some trading strategy $X_t^Y$, we can write

$$Y = a + X_T^Y$$

where $a$ is the Black Scholes price of the put. Now the right hand side of equation (5.3) can be written as

$$\max_{\{X_t\}} \mathbb{E}[U(x_0 + p_\varepsilon + X_T - \varepsilon a - \varepsilon X_T^Y)]$$

and since all trading strategies form a linear space, taking the trading strategy $\varepsilon X_t^Y$ from the strategy $X_t$, leaves another trading strategy, $\tilde{X}_t$ say, and the equation for $p_\varepsilon$ becomes

[1]See Eeckhoudt and Gollier (1995), Chapter 4, for a discussion of bid and ask prices.

$$\max_{\{X_t\}} \mathbb{E}[U(x_0 + X_T)] = \max_{\{\tilde{X}_t\}} \mathbb{E}[U(x_0 + (p_\varepsilon - \varepsilon a) + \tilde{X}_T)].$$

Maximising over $\tilde{X}_t$ is the same as maximising over $X_t$ and so it must be the case that $p_\varepsilon = \varepsilon a$.

In our time-lagged economy, however, the Black-Scholes formula does not apply and it will not necessarily be the case that $p_\varepsilon = p_\varepsilon^{\mathrm{b}}$. The maximum price the investor is willing to pay for $\varepsilon$ puts, is less than or equal to the minimum price he is willing to accept for selling $\varepsilon$ puts, i.e. $p_\varepsilon^{\mathrm{b}} \leq p_\varepsilon$. We are interested not in establishing any one price for the put option, but rather in discovering the effect of a time-lag on a buyer, or a seller of the put. We therefore concentrate on calculating the fair price to the seller (or to the buyer) of $\varepsilon$ put options in both the lag and no-lag economies. We look for the moment at the seller's price. To do this we need to solve the problem

$$\max \mathbb{E}[U(w_T)|w_0]$$

over all possible trading strategies in the risky asset where a time $t$ trade must be precommitted to at time $t - \delta$ for small $\delta > 0$. We look specifically at the problem of an investor with utility function given by

$$U(x) = -\mathrm{e}^{-\gamma x},$$

where $\gamma > 0$ is a constant.

We start by using a binomial approximation of the share price process as described in Chapter 4.

## 5.2 Discrete-time method

Time to expiry $T$ is divided into $N$ equal steps of length $T/N = h$. The investor can rebalance his portfolio only at times $0, h, 2h, \ldots$ and the time-lag is also assumed to be of length $h$. Therefore at time $nh$ the investor commits to holding some amount $\theta_{n+1}$ in the risky asset over the time period $((n+1)h, (n+2)h]$. The remainder of his wealth is invested in the riskless asset (or bank account). This amount can be negative indicating borrowing or short selling of the riskless asset.

We calculate the value function, $V_n(w, \theta, s)$, at time $nh$ where $w_n = w$ is the wealth of the investor at time $nh$; $\theta_n = \theta$ and $s_n = s$ is the price of the risky asset at time $nh$. This is given by

$$V_n(w, \theta, s) = \max_{\theta_{n+1}} \mathbb{E}[U(w_N - Y_{s_N})|w_n = w, \theta_n = \theta, s_n = s]$$

where $Y_s = (K - s)^+$ is the payoff of the put option if the share price at time $T$ is $s$. We can re-express $V_n$ as

$$\begin{aligned} V_n(w, \theta, s) \quad = \quad & \max_{\theta_{n+1}} [pV_{n+1}(\rho w + \theta(a - \rho), \theta_{n+1}, sa) \\ & \qquad + (1 - p)V_{n+1}(\rho w + \theta(1/a - \rho), \theta_{n+1}, s/a)] \end{aligned}$$

since if the share price moves from $s$ to $sa$ (with probability $p$) the wealth of the investor becomes $\theta a + (w - \theta)\rho$ and if the share price moves down to $s/a$ with probability $1 - p$ the wealth becomes $\theta/a + (w - \theta)\rho$.

To simplify the problem, we can write

$$V_n(w,\theta,s) \equiv \mathrm{e}^{-\gamma_n w} g_n(\theta,s) \tag{5.4}$$

for some function $g_n(\theta, s)$, where $\gamma_n = \gamma\rho^{N-n}$. This is shown by induction with final step

$$\begin{aligned}
V_{N-1}(w,\theta,s) &= \mathbb{E}[U(w_N - Y_{s_N})|w_{N-1} = w, \theta_{N-1} = \theta, s_{N-1} = s] \\
&= pU(\rho w + \theta(a-\rho) - Y_{sa}) + (1-p)U(\rho w + \theta(1/a - \rho) - Y_{s/a}) \\
&= \mathrm{e}^{-\gamma\rho w}[pU(\theta(a-\rho) - Y_{sa}) + (1-p)U(\theta(1/a-\rho) - Y_{s/a})] \\
&= \mathrm{e}^{-\gamma\rho w} g_{N-1}(\theta,s)
\end{aligned}$$

and for the inductive step, assuming equation (5.4) holds for $n \geq m+1$

$$\begin{aligned}
V_m(w,\theta,s) &= \max_{\theta_{m+1}} [p\mathrm{e}^{-\gamma\rho^{N-m-1}(\rho w + \theta(a-\rho))} g_{m+1}(\theta_{m+1}, sa) \\
&\qquad +(1-p)\mathrm{e}^{-\gamma\rho^{N-m-1}(\rho w + \theta(1/a-\rho))} g_{m+1}(\theta_{m+1}, s/a)] \\
&= \mathrm{e}^{-\gamma\rho^{N-m} w} \max_{\theta_{m+1}} [p\mathrm{e}^{-\gamma\rho^{N-m-1}\theta(a-\rho)} g_{m+1}(\theta_{m+1}, sa) \\
&\qquad +(1-p)\mathrm{e}^{-\gamma\rho^{N-m-1}\theta(1/a-\rho)} g_{m+1}(\theta_{m+1}, s/a)] \\
&= \mathrm{e}^{-\gamma_m w} g_m(\theta, s).
\end{aligned}$$

We now have an algorithm for $g$ given by

$$g_{N-1}(\theta, s) = pU(\theta(a-\rho) - Y_{sa}) + (1-p)U(\theta(1/a-\rho) - Y_{s/a})$$

and

$$g_n(\theta, s) = \max_{\theta_{n+1}} [pe^{-\gamma_{n+1}\theta(a-\rho)} g_{n+1}(\theta_{n+1}, sa) + (1-p)e^{-\gamma_{n+1}\theta(1/a-\rho)} g_{n+1}(\theta_{n+1}, s/a)]$$

$\forall n = N-2, ..., 0$.

The difficulty is that each maximising value of $\theta_{n+1}$, denoted by $\theta^*_{n+1}$, depends on $\theta$ and $s$ so that $g_n(\theta, s)$ can only be found at a discrete set of values of $\theta$. We chose a range wide enough to include $\theta^*$ at each step (the range was found by trial and error simply by making sure that the $\theta^*$ found was not at either end of the range). Then this range was divided into $M$ equal lengths and the function was maximised over $\theta$ taking values only at these discrete points. The maximisation was done by golden section search in order to speed up the process.

### 5.2.1 Maximisation by Golden Section

The golden section search is based on dividing the search interval into three pieces with the points between the pieces chosen so that the distance between each point and the furthest end of the search interval is a specific value. This value is known as the golden proportion. To construct the golden section we take four points, one at either end of the search range and two at specified points within it, dividing the range into three intervals. Since the function to be maximised is concave (due to the concavity of the utility function) it has no local maxima apart from the global maximum. It is therefore possible to eliminate one of the two outer intervals from the search by looking at the value of the function at the two central points. If the first is larger than the second, the interval on the right can be eliminated; and if the second is larger, that on the left can be eliminated.

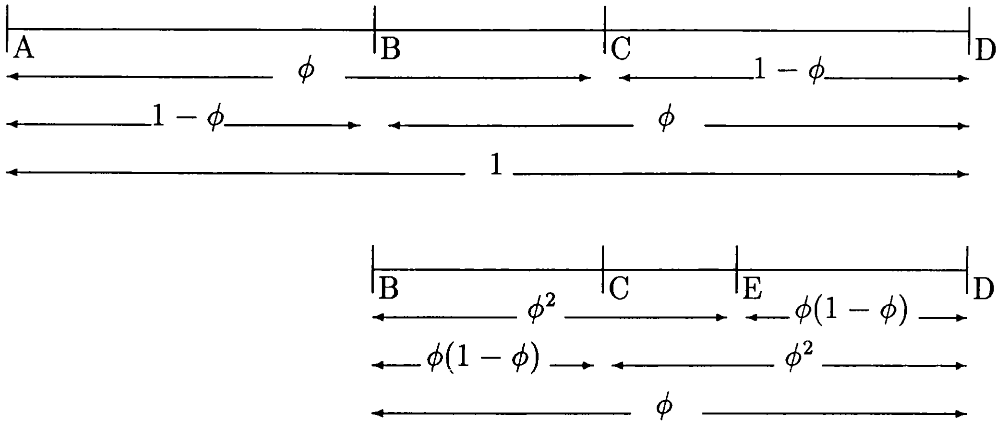


Figure 5-1: The Golden Section

Letting AD in Figure 5-1 be of length one, and then taking AB to be $1-\phi$ and AC to be $\phi$, this leaves BD as $\phi$ and CD as $1-\phi$. Now if AB is eliminated, leaving only the segment BD shown in the second part of the diagram, we require C to be a fraction $1-\phi$ along BD. This makes the distance BC equal to $\phi(1-\phi)$. In order for this to be true, $\phi$ must be chosen so that firstly, from the top part of the diagram

$$
\begin{aligned}
\mathrm{BC} &= \mathrm{AD} - \mathrm{AB} - \mathrm{CD} \\
&= 1 - (1-\phi) - (1-\phi) = 2\phi - 1
\end{aligned}
$$

and from the second part of the diagram

$$\mathrm{BC} = \phi(1-\phi)$$

so $\phi$ must solve

$$2\phi - 1 = \phi - \phi^2.$$

Now a new point E is chosen so that $\frac{\text{BE}}{\text{BD}} = \phi$ making BE $= \phi^2$.

Clearly, with this method, the value of the function to be maximised will be required at other values of $\theta$ than simply the discrete ones available to us. It is therefore necessary to interpolate between discrete points. Once two or more of the required points fall within one step, there is no point in doing any more than maximising over the discrete points which remain. There will be no more than five such points left at this stage.

In this way we run the algorithm backwards to time 0, obtaining

$$g_0(\theta, S_0)$$

for $\theta$ taking values at the discrete points. We now maximise this over $\theta$, calling this maximising value $\theta^*$. The solution to the maximisation problem is given by

$$\begin{aligned}
\max\mathbb{E}[U(w_N - \varepsilon Y)|w_0 = w] &= V_0(w, \theta^*, S_0) \\
&= \mathrm{e}^{-\gamma w \rho^N} g_0(\theta^*, S_0) \\
&\equiv \mathrm{e}^{-\gamma w \rho^N} u_Y.
\end{aligned}$$

Repeating the calculations using $Y = 0$ gives

$$u \equiv \max\mathbb{E}[U(w_N)|w_0 = 0] = \tilde{g}_0(\theta^{**}, S_0)$$

where $\theta^{**}$ maximises $\tilde{g}_0(\theta, S_0)$ and $\tilde{g}$ is the algorithm when $Y = 0$. From equation (5.1) with $x_0 = 0$, the price $p_\varepsilon$ of $\varepsilon$ put options now satisfies

$$\mathrm{e}^{-\gamma p_\varepsilon \rho^N} u_Y = u$$

that is

$$p_\varepsilon = \frac{1}{\gamma\rho^N} \ln\left(\frac{u_Y}{u}\right).$$

### 5.2.2 No time-lag case

In order to test the methods used in the program for the time-lag case, we also use the method when there is no lag and compare the price obtained with the usual binomial tree method and with the Black Scholes price.

In this case, the algorithm is

$$\begin{aligned}
V_n(w,s) &= \max_{\theta_n} \mathrm{E}[U(w_N - Y_{s_N})|w_n = w, s_n = s] \\
&= \max_{\theta_n} [pV_{n+1}(\rho w + \theta_n(a-\rho), sa) + (1-p)V_{n+1}(\rho w + \theta_n(1/a - \rho), s/a)] \\
&= \max_{\theta_n} [p\mathrm{e}^{-\gamma_{n+1}(\rho w + \theta_n(a-\rho))} g_{n+1}(sa) + (1-p)\mathrm{e}^{-\gamma_{n+1}(\rho w + \theta_n(1/a-\rho))} g_{n+1}(s/a)] \\
&= \mathrm{e}^{-\gamma_n w} \max_{\theta_n} [p\mathrm{e}^{-\gamma_{n+1}\theta_n(a-\rho)}\alpha + (1-p)\mathrm{e}^{-\gamma_{n+1}\theta_n(1/a-\rho)}\beta]
\end{aligned}$$

where $\alpha = g_{n+1}(sa)$, $\beta = g_{n+1}(s/a)$ and where we use the identity $V_n(w,s) = \mathrm{e}^{-\gamma_n w} g_n(s)$ obtained similarly to that used in the time-lag case.

At time step $N$ we have

$$V_N(w,s) = [U(w_N - Y_{s_N})|w_N = w, s_N = s]$$

$$\begin{aligned} &= -e^{-\gamma(w-Y_s)} \\ &= e^{-\gamma w}(-e^{\gamma Y_s}) \end{aligned}$$

and so the simplified algorithm is

$$g_N(s) = -e^{\gamma Y_s}$$

and for $n < N$

$$g_n(s) = \max_{\theta_n} \, [pe^{-\gamma_{n+1}\theta_n(a-\rho)}\alpha + (1-p)e^{-\gamma_{n+1}\theta_n(1/a-\rho)}\beta].$$

So for each $n$ we need to maximise

$$f(\eta) \equiv pe^{-\gamma_{n+1}\eta(a-\rho)}\alpha + (1-p)e^{-\gamma_{n+1}\eta(1/a-\rho)}\beta \tag{5.5}$$

where $\alpha$ and $\beta$ are constant with respect to $\eta$. Since $f$ is concave we differentiate to find the turning point $\eta^*$

$$\eta^* = \frac{1}{\gamma_{n+1}(a-1/a)} \ln \left[ \frac{p\alpha(a-\rho)}{(1-p)\beta(\rho-1/a)} \right]$$

and the function $g_0(s)$ is calculated by working back through the tree. The analysis for the bid price of $\varepsilon$ put options is done in exactly the same way as for the ask price, replacing $\varepsilon$ with $-\varepsilon$. The resulting $p_{-\varepsilon}$ would then be negative (indicating that it is being paid out at time 0 instead of received) and equal to $-p^{\mathrm{b}}_{\varepsilon}$. In Chapter 6 we give the results for the discrete-time case. For Example 1 the results for both the buyer and the seller are given and compared.

## 5.3 Continuous Case

We start, as in the discrete-time case, with the share price process $S_t$ defined by

$$\mathrm{d}S_t = S_t(\sigma \mathrm{d}W_t + \mu \mathrm{d}t),$$

or

$$S_t = S_0 \exp\left[\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right) t\right]$$

for constants $S_0$, $\mu$ and $\sigma$.

### 5.3.1 No time-lag, no payout

Our first problem is to calculate the basic value function for maximising final utility with no payout and no time-lag. We call this $V_I$, given by

$$V_{\mathrm{I}}(x_0) \equiv \max \mathbb{E}[U(x_0 \mathrm{e}^{rT} + X_T)] \tag{5.6}$$

where $x_0$ is initial wealth, $T$ is the time horizon and $x_0\mathrm{e}^{rt} + X_t$ is wealth at time $t$. Maximisation takes place over all possible wealth processes gained from different investment strategies. Suppose we hold $\$\theta_t$ in shares at time $t$. Then the wealth process $X_t$ gained from this investment strategy satisfies

$$\begin{aligned}
\mathrm{d}X_t &= \theta_t \frac{\mathrm{d}S_t}{S_t} + (X_t - \theta_t) r \mathrm{d}t \\
&= \theta_t(\sigma \mathrm{d}W_t + \mu \mathrm{d}t) + (X_t - \theta_t) r \mathrm{d}t \\
&= rX_t \mathrm{d}t + \theta_t(\sigma \mathrm{d}W_t + (\mu - r)\mathrm{d}t)
\end{aligned}$$

$$\Rightarrow \qquad \mathrm{d}(\mathrm{e}^{-rt}X_t) = \mathrm{e}^{-rt}\mathrm{d}X_t - r\mathrm{e}^{-rt}X_t \mathrm{d}t$$

$$= \quad e^{-rt}\{\theta_t(\sigma dW_t + (\mu - r)dt)\}.$$

Now if we let $Z_t$ be defined by

$$\frac{dZ_t}{Z_t} = -\frac{\mu - r}{\sigma} dW_t,$$

i.e.

$$Z_t = \exp\left[-\left(\frac{\mu - r}{\sigma}\right) W_t - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 t\right],$$

then, by the Cameron-Martin-Girsanov theorem, under the measure $\mathbb{P}^*$ defined by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = Z_T,$$

$W_t^*$ is Brownian motion where

$$W_t^* = W_t + \frac{\mu - r}{\sigma} t.$$

So now we can write

$$d(e^{-rt} X_t) = e^{-rt} \theta_t \sigma dW_t^*$$

or

$$X_t = e^{rt} \int_0^t e^{-rs} \theta_s \sigma dW_s^*.$$

**Lemma 5.3.1** *If the maximum in equation (5.6) is achieved with wealth process $X^*$ and investment process $\theta^*$, then first order conditions say that*

$$U'(x_0 e^{rT} + X_T^*) \propto \zeta_T$$

*where* $\zeta_t = e^{-rt} Z_t$.

**Proof** (Correct to first order in $\varepsilon$.)

Since $\theta^*$ is optimal, if we perturb $\theta_t^* \mapsto \theta_t^* + \varepsilon \eta_t$, for any process $\eta_t$, then the payoff will not be increased, i.e.

$$\begin{aligned} 0 &\leq \mathbb{E}[U(x_0 e^{rT} + X_T^*) - U(x_0 e^{rT} + X_T^* + e^{rT} \int_0^T e^{-rs} \varepsilon \eta_s \sigma \mathrm{d}W_s^*)] \\ &\simeq -\mathbb{E}[U'(x_0 e^{rT} + X_T^*) e^{rT} \int_0^T \sigma e^{-rs} \varepsilon \eta_s \mathrm{d}W_s^*]. \end{aligned}$$

By changing $\varepsilon$ to $-\varepsilon$ we can deduce that

$$\mathbb{E}[U'(x_0 e^{rT} + X_T^*) e^{rT} \int_0^T H_s \mathrm{d}W_s^*] = 0 \ \forall \ H \tag{5.7}$$

Therefore $W^*$ is a martingale under the measure $\hat{\mathbb{P}}$ where

$$\frac{\mathrm{d}\hat{\mathbb{P}}}{\mathrm{d}\mathbb{P}} \propto U'(x_0 e^{rT} + X_T^*) e^{rT}.$$

Since $W^*$ is also a martingale under $\mathbb{P}^*$ and $\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}} = Z_T$, we must have, in this case where we have one-dimensional Brownian motion,

$$e^{rT} U'(x_0 e^{rT} + X_T^*) \propto Z_T = e^{rT} \zeta_T.$$

We can therefore write, for any $x_0$,

$$U'(x_0 e^{rT} + X_T^*) = \lambda(x_0)\zeta_T \tag{5.8}$$

for some deterministic function $\lambda$. □

Another consequence of this result is that, for any process $H$

$$\begin{aligned} \mathbb{E}[U'(x_0 e^{rT} + X_T^*)H] &= \mathbb{E}[\lambda(x_0)e^{-rT} Z_T H] \\ &= \mathbb{E}^*[\lambda(x_0)e^{-rT} H]. \end{aligned} \tag{5.9}$$

Returning to our basic problem, we can write the value function as

$$\begin{aligned} V_{\mathrm{I}}(x_0) &\equiv \mathbb{E}[U(x_0 e^{rT} + X_T^*)] \\ &= \frac{-1}{\gamma}\mathbb{E}[U'(x_0 e^{rT} + X_T^*)] \\ &= -\frac{\lambda(x_0)}{\gamma}\mathbb{E}[\zeta_T] \\ &= -\frac{\lambda(x_0)}{\gamma}e^{-rT}. \end{aligned}$$

To calculate $\lambda(x_0)$, from equation (5.8)

$$\begin{aligned} x_0 e^{rT} + X_T^* &= (U')^{-1}(\lambda(x_0)\zeta_T) \\ &= -\frac{1}{\gamma}\ln\frac{\lambda(x_0)}{\gamma} + \frac{1}{\gamma}\left[\frac{\mu - r}{\sigma}W_T^* + \left(r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right)T\right]. \end{aligned} \tag{5.10}$$

Taking $\mathbb{E}^*$ of both sides gives

$$x_0 e^{rT} = -\frac{1}{\gamma}\ln\frac{\lambda(x_0)}{\gamma} + \frac{1}{\gamma}\left(r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right)T$$

$$\Rightarrow \qquad \lambda(x_0) = \gamma \exp\left[\left(r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right)T - \gamma x_0 e^{rT}\right]. \tag{5.11}$$

Substituting this into equation (5.10) gives

$$X_T^* = \frac{\mu - r}{\gamma\sigma}W_T^* = \int_0^T \frac{\mu - r}{\gamma\sigma}\mathrm{d}W_s^*$$

and since we also know that

$$X_T^* = \int_0^T e^{r(T-s)}\theta_s^*\sigma \mathrm{d}W_s^*,$$

the optimal portfolio must be

$$\theta_s^* = e^{-r(T-s)}\frac{\mu - r}{\gamma\sigma^2}. \tag{5.12}$$

### 5.3.2 Time-lag with no payout

Now let $V_{II}$ be the value function when there is a trading time-lag of length $\delta$ but still no payout. Then

$$V_{II}(x_0) = \max\left\{\mathbb{E}\left[U\left(x_0 e^{rT} + \int_0^T e^{r(T-s)}\theta_s\sigma \mathrm{d}W_s^*\right)\right] : \theta_s \in \mathcal{F}_{s-\delta}\right\}.$$

Since the $\theta_s$ which achieves the maximal value of $V_I(x_0)$ is deterministic, it satisfies $\theta_s \in \mathcal{F}_{s-\delta}$ and therefore it must be the case that

$$V_{II}(x_0) = V_I(x_0).$$

### 5.3.3 Time-lag with payout

To find the seller's price for $\varepsilon$ put options, we need to find

$$V_{III}(x_0,\varepsilon) = \max\left\{\mathbb{E}\left[U\left(x_0\mathrm{e}^{rT} + \int_0^T \mathrm{e}^{r(T-s)}\theta_s\sigma\mathrm{d}W^*_s - \varepsilon Y\right)\right] : \theta_s \in \mathcal{F}_{s-\delta}\right\}$$

where $Y = (K - S_T)^+$. Writing the portfolio process as $\theta^{III}_s = \theta^*_s + \psi_s$ and noting that for small $\varepsilon$ and $\delta$, $\psi_s$ and $\varepsilon Y$ are small, we can apply Taylor's theorem. We also make the assumption for the moment that $\psi_s$ is of order $\varepsilon$ as it simplifies the expansion. Now the problem is to maximise

$$\begin{aligned}
&\mathbb{E}\left[U\left(x_0\mathrm{e}^{rT} + \int_0^T \mathrm{e}^{r(T-s)}\sigma\{\theta^*_s + \psi_s\}\mathrm{d}W^*_s - \varepsilon Y\right)\right]\\
\simeq\ &\mathbb{E}\left[U(x_0\mathrm{e}^{rT} + X^*_T) + U'(x_0\mathrm{e}^{rT} + X^*_T)\left\{\int_0^T \sigma\mathrm{e}^{r(T-s)}\psi_s\mathrm{d}W^*_s - \varepsilon Y\right\}\right.\\
&\qquad\left. + \frac{1}{2}U''(x_0\mathrm{e}^{rT} + X^*_T)\left\{\int_0^T \sigma\mathrm{e}^{r(T-s)}\psi_s\mathrm{d}W^*_s - \varepsilon Y\right\}^2\right]\\
=\ &V_I(x_0) + \lambda(x_0)\mathrm{e}^{-rT}\mathbb{E}^*[-\varepsilon Y] - \frac{1}{2}\lambda(x_0)\gamma\mathrm{e}^{-rT}\mathbb{E}^*\left\{\int_0^T \sigma\mathrm{e}^{r(T-s)}\psi_s\mathrm{d}W^*_s - \varepsilon Y\right\}^2.
\end{aligned}$$

We can replicate the payoff of the put option with a portfolio process, $\theta^Y_s$

$$y_t = a\mathrm{e}^{rt} + \int_0^t \sigma\mathrm{e}^{r(t-s)}\theta^Y_s\mathrm{d}W^*_s$$

where $a$ is the Black-Scholes price of the put and $y_T = Y = (K - S_T)^+$ so that the problem is now to minimise (for $\theta^{III}_s \in \mathcal{F}_{s-\delta}$)

$$\begin{aligned}
&\mathbb{E}^*\left[\int_0^T \sigma e^{r(T-s)}(\theta_s^{III}-\theta_s^*-\varepsilon\theta_s^Y)dW_s^* - \varepsilon a e^{rT}\right]^2 \\
&= \mathbb{E}^*\left(\int_0^T \sigma^2 e^{2r(T-s)}(\theta_s^{III}-\theta_s^*-\varepsilon\theta_s^Y)^2 ds\right) + \varepsilon^2 a^2 e^{2rT} \qquad (5.13)
\end{aligned}$$

where here we make the assumption that $(\theta_s^{III}-\theta_s^*-\varepsilon\theta_s^Y)$ is well behaved enough to satify the Ito isometry which says that

$$\mathbb{E}\left[\int f_s dW_s\right]^2 = \mathbb{E}\left[\int f_s^2 ds\right]$$

for a certain class of functions $f_s$.

To minimise (5.13) we must take[2]

$$\begin{aligned}
\theta_s^{III} &= \mathbb{E}^*[\theta_s^* + \varepsilon\theta_s^Y|\mathcal{F}_{s-\delta}] \\
&= \theta_s^* + \varepsilon\mathbb{E}^*(\theta_s^Y|\mathcal{F}_{s-\delta})
\end{aligned}$$

which gives us, as required, $\psi_s$ to be of order $\varepsilon$ and therefore

$$V_{III}(x_0,\varepsilon) \simeq V_I(x_0) - \lambda(x_0)\varepsilon a - \frac{1}{2}\gamma\lambda(x_0)e^{-rT}\left[\varepsilon^2 a^2 e^{2rT} + \varepsilon^2 A\right]$$

where

$$A = \mathbb{E}^*\int_0^T \sigma^2 e^{2r(T-s)}\left[\mathbb{E}^*(\theta_s^Y|\mathcal{F}_{s-\delta}) - \theta_s^Y\right]^2 ds.$$

[2]See for example Williams(1991), page 85.

### 5.3.4 Calculating the Price

Rewriting equation (5.1) for the marginal price of a put in the notation of this section,

$$V_{III}(x_0 + p(\varepsilon), \varepsilon) = V_{II}(x_0) \tag{5.14}$$

gives us $p(\varepsilon)$, the price of $\varepsilon$ puts in the time-lagged economy. From our previous calculations, the left hand side of this equation is, to order $\varepsilon^2$,

$$\begin{aligned} & V_I(x_0 + p(\varepsilon)) - \lambda(x_0 + p(\varepsilon))\varepsilon a - \frac{1}{2}\gamma\lambda(x_0 + p(\varepsilon))\mathrm{e}^{-rT}\left[\varepsilon^2 a^2 \mathrm{e}^{2rT} + \varepsilon^2 A\right] \\ = \; & -\lambda(x_0 + p(\varepsilon))\left[\frac{1}{\gamma}\mathrm{e}^{-rT} + \varepsilon a + \frac{1}{2}\gamma\mathrm{e}^{-rT}\varepsilon^2(a^2\mathrm{e}^{2rT} + A)\right] \end{aligned}$$

and the right hand side is

$$V_I(x_0) = -\frac{\lambda(x_0)}{\gamma}\mathrm{e}^{-rT}.$$

Substituting for $\lambda(x_0)$ from equation (5.11)

$$\begin{aligned} \mathrm{e}^{\gamma p(\varepsilon)\mathrm{e}^{rT}} &= 1 + \varepsilon a\gamma\mathrm{e}^{rT} + \frac{1}{2}\gamma^2\varepsilon^2(a^2\mathrm{e}^{2rT} + A) \\ \Rightarrow p(\varepsilon) &= \frac{1}{\gamma\mathrm{e}^{rT}}\ln\left[1 + \varepsilon a\gamma\mathrm{e}^{rT} + \frac{1}{2}\gamma^2\varepsilon^2(a^2\mathrm{e}^{2rT} + A)\right]. \end{aligned} \tag{5.15}$$

It remains only to calculate $A$. We know that

$$A = \mathbb{E}^*\int_0^T \sigma^2\mathrm{e}^{2r(T-s)}(\mathbb{E}^*(\theta_s^Y|\mathcal{F}_{s-\delta}) - \theta_s^Y)^2\mathrm{d}s \tag{5.16}$$

and must now calculate the replicating portfolio, $\theta_s^Y$, for the put using

$$y_t = a\mathrm{e}^{rt} + \int_0^t \sigma \mathrm{e}^{r(t-s)} \theta_s^Y \mathrm{d}W_s^*. \tag{5.17}$$

$\tilde{y}_t = \mathrm{e}^{-rt} y_t$ is a martingale with $\tilde{y}_0 = a$;

$$\mathrm{d}(\tilde{y}_t) = \mathrm{e}^{-rt} \theta_t^Y \sigma \mathrm{d}W_t^*; \tag{5.18}$$

and we can also write

$$\begin{aligned} \tilde{y}_t &= \mathbb{E}^*[\mathrm{e}^{-rT} g(W_T^*)|\mathcal{F}_t] \\ &= \mathrm{e}^{-rT} P_{T-t} g(W_t^*) \end{aligned} \tag{5.19}$$

where $g(W_T^*)$ is the payoff of the put and $P_t$ is the transition semi-group of Brownian motion. So now we have

$$\begin{aligned} \mathrm{d}\tilde{y}_t &= \mathrm{e}^{-rT} (\nabla P_{T-t} g)(W_t^*) \mathrm{d}W_t^* \\ &= \mathrm{e}^{-rT} (P_{T-t} \nabla g)(W_t^*) \mathrm{d}W_t^*. \end{aligned} \tag{5.20}$$

(For a proof of the last line see Appendix, Section 8.2.1.)

Comparing equations (5.18) and (5.20), we can write

$$\theta_t^Y = \frac{1}{\sigma \mathrm{e}^{-rt}} \mathrm{e}^{-rT} (P_{T-t} \nabla g)(W_t^*)$$

and also

$$\mathbb{E}^*[\theta_t^Y | \mathcal{F}_{T-\delta}] = \frac{1}{\sigma} \mathrm{e}^{-r(T-t)} \mathbb{E}^*[(P_{T-t} \nabla g)(W_t^*) | \mathcal{F}_{t-\delta}]$$

$$= \frac{1}{\sigma}\mathrm{e}^{-r(T-t)}(P_{(T-t+\delta)\wedge T}\nabla g)(W^*_{(t-\delta)^+}).$$

(Again, see Appendix, Section 8.2.1 for a proof of the last line.)

Substituting into equation (5.16), we have

$$\begin{aligned}
A &= \mathbb{E}^*\int_0^T(\mathbb{E}^*[(P_{T-s}\nabla g))(W^*_s)|\mathcal{F}_{s-\delta}] - (P_{T-s}\nabla g)(W^*_s))^2\mathrm{d}s \\
&= \mathbb{E}^*\int_0^T(P_{T-s}\nabla g)(W^*_s)^2 - (\mathbb{E}^*[(P_{T-s}\nabla g)(W^*_s)|\mathcal{F}_{s-\delta}])^2\mathrm{d}s \\
&= \mathbb{E}^*\int_0^T\{(P_{T-s}\nabla g)(W^*_s)^2 - (P_{(T-s+\delta)\wedge T}\nabla g)(W^*_{(s-\delta)^+})^2\}\mathrm{d}s \\
&= \mathbb{E}^*\int_{T-\delta}^T(P_{T-s}\nabla g)(W^*_s)^2\mathrm{d}s - \delta(P_T\nabla g)(W^*_0)^2 \\
&= \mathbb{E}^*(M^2_T - M^2_{T-\delta}) - \delta(P_T\nabla g)(0)^2 \qquad (5.21)
\end{aligned}$$

where $M^2_t = \int_0^t(P_{T-s}\nabla g)(W^*_s)^2\mathrm{d}s$. To calculate $M^2_t$ we first write equation (5.20) as

$$\begin{aligned}
\int_0^t(P_{T-s}\nabla g)(W^*_s)\mathrm{d}W^*_s &= \mathrm{e}^{rT}\int_0^t\mathrm{d}\tilde{y}_s \\
&= \mathrm{e}^{rT}(y_t\mathrm{e}^{-rt} - a)
\end{aligned}$$

and note, from the Black-Scholes theorem, that

$$\mathrm{e}^{-rt}y_t = K\mathrm{e}^{-rT}\Phi(-d^t_2) - S_t\mathrm{e}^{-rt}\Phi(-d^t_1) \qquad (5.22)$$

where

$$d^t_1 = \frac{\ln(S_t/K) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$
\begin{aligned}
d_2^t &= d_1 - \sigma\sqrt{T-t} \\
\Phi(x) &= \int_{-\infty}^{x} \frac{\mathrm{e}^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \mathrm{d}u.
\end{aligned}
$$

So we have

$$
\begin{aligned}
\mathbb{E}^* M_T^2 &= \mathbb{E}^* \int_0^T (P_{T-t}\nabla g)(W_t^*)^2 \mathrm{d}t \\
&= \mathbb{E}^* \left[ \int_0^T (P_{T-t}\nabla g)(W_t^*) \mathrm{d}W_t^* \right]^2 \\
&= \mathbb{E}^* [Y - a\mathrm{e}^{rT}]^2 \\
&= \mathbb{E}^* Y^2 - a^2 \mathrm{e}^{2rT} \\
&= \mathbb{E}^* ((K - S_T)^2 I_{K>S_T}) - a^2 \mathrm{e}^{2rT} \\
&= K^2 \mathbb{P}^*(K > S_T) - 2K\mathbb{E}^*(S_T I_{K>S_T}) + \mathbb{E}^*(S_T^2 I_{K>S_T}) - a^2 \mathrm{e}^{2rT}.
\end{aligned}
$$

The second line relies on the fact that $(P_{T-t}\nabla g)(W_t^*)$ is bounded because $\theta_t^Y$ is bounded and the following identities are proved in the Appendix, Section 8.2.2

$$
\begin{aligned}
\mathbb{P}^*(K > S_T) &= \Phi(-d_2^0) \\
\mathbb{E}^*(S_T I_{K>S_T}) &= S_0 \mathrm{e}^{rT} \Phi(-d_1^0) \\
\mathbb{E}^*(S_T^2 I_{K>S_T}) &= S_0^2 \mathrm{e}^{(2r+\sigma^2)T} \Phi\left( \frac{1}{\sigma\sqrt{T}} \left\{ \ln\frac{K}{S_0} - \left( r + \frac{3}{2}\sigma^2 \right) T \right\} \right).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\mathbb{E}^* M_{T-\delta}^2 &= \mathbb{E}^* \{ \mathrm{e}^{rT} (y_{T-\delta} \mathrm{e}^{-r(T-\delta)} - a) \}^2 \\
&= \mathbb{E}^* \{ K\Phi(-d_2^{T-\delta}) - S_{T-\delta} \mathrm{e}^{r\delta} \Phi(-d_1^{T-\delta}) - a\mathrm{e}^{rT} \}^2 \\
&= K^2 \mathbb{E}^* [\Phi^2(-d_2^{T-\delta})] + \mathrm{e}^{2r\delta} \mathbb{E}^* [\Phi^2(-d_1^{T-\delta}) S_{T-\delta}^2] + a^2 \mathrm{e}^{2rT} \\
&\qquad - 2K\mathrm{e}^{r\delta} \mathbb{E}^* [S_{T-\delta} \Phi(-d_2^{T-\delta}) \Phi(-d_1^{T-\delta})]
\end{aligned}
$$

$$- 2Kae^{rT}\mathbb{E}^*[\Phi(-d_2^{T-\delta})] + 2ae^{r(T+\delta)}\mathbb{E}^*[S_{T-\delta}\Phi(-d_1^{T-\delta})]$$

where $\Phi^2(.)$ is used to signify $[\Phi(.)]^2$. This can be evaluated by numerical integration. For example

$$\mathbb{E}^*\Phi^2(-d_2^{T-\delta}) = \int_{-\infty}^{\infty} \Phi^2(f(y))\mathbb{P}^*(W_{T-\delta}^* \in \mathrm{d}y)$$

where

$$f(y) = \frac{\ln\frac{K}{S_0} - \left[\sigma y + \left(r - \frac{1}{2}\sigma^2\right)(T-\delta)\right] - \left(r - \frac{1}{2}\sigma^2\right)\delta}{\sigma\sqrt{\delta}}$$

and

$$\mathbb{P}^*(W_{T-\delta}^* \in \mathrm{d}y) = \frac{\exp\left[-\frac{y^2}{2(T-\delta)}\right]}{\sqrt{2\pi(T-\delta)}}\mathrm{d}y.$$

Lastly, since $e^{-rt}y_t$ is a martingale, we have

$$\frac{\partial}{\partial W^*}(e^{-rt}y_t) = e^{-rt}\theta_t^Y\sigma,$$

and differentiation of equation (5.22) gives

$$\theta_0^Y = -\Phi(-d_1^0)S_0.$$

Therefore

$$\begin{aligned}(P_T\nabla g)(0) &= e^{rT}\theta_0^Y\sigma \\ &= -\sigma e^{rT}S_0\Phi(-d_1^0).\end{aligned} \tag{5.23}$$

### 5.3.5 Formulae

Combining equations (5.15), (5.21) and (5.23), the price, $p(\varepsilon)$, for $\varepsilon$ puts is

$$p(\varepsilon) = \frac{1}{\gamma e^{rT}} \ln \left\{ 1 + \varepsilon a \gamma e^{rT} + \frac{1}{2}\gamma^2 \varepsilon^2 \left[ a^2 e^{2rT} + \mathbb{E}^* M_T^2 - \mathbb{E}^* M_{T-\delta}^2 - \delta \sigma^2 e^{2rT} S_0^2 \Phi^2(-d_1^0) \right] \right\}$$

where

$$\mathbb{E}^* M_T^2 = K^2 \Phi(-d_2^0) - 2K S_0 e^{rT} \Phi(-d_1^0) + S_0^2 e^{(2r+\sigma^2)T} \Phi(\bar{d}_1^0) - a^2 e^{2rT}$$

where

$$\bar{d}_1^0 = \frac{1}{\sigma\sqrt{T}} \left\{ \ln \frac{K}{S_0} - \left( r + \frac{3}{2}\sigma^2 \right) T \right\}$$

and where

$$\mathbb{E}^* M_{T-\delta}^2 = K^2 I_1 + e^{2r\delta} I_2 + a^2 e^{2rT} - 2aK e^{rT} I_3 - 2K e^{r\delta} I_4 + 2a e^{r(T+\delta)} I_5$$

where the integrals $I_1$, $I_2$, ..., $I_5$ are calculated numerically from the formulae

$$\begin{aligned}
I_1 &= \int_{-\infty}^{\infty} \Phi^2(f(y)) \mathbb{P}^*(W_{T-\delta}^* \in \mathrm{d}y) \\
I_2 &= \int_{-\infty}^{\infty} (s(y))^2 \Phi^2(g(y)) \mathbb{P}^*(W_{T-\delta}^* \in \mathrm{d}y) \\
I_3 &= \int_{-\infty}^{\infty} \Phi(f(y)) \mathbb{P}^*(W_{T-\delta}^* \in \mathrm{d}y) \\
I_4 &= \int_{-\infty}^{\infty} s(y) \Phi(f(y)) \Phi(g(y)) \mathbb{P}^*(W_{T-\delta}^* \in \mathrm{d}y) \\
I_5 &= \int_{-\infty}^{\infty} s(y) \Phi(g(y)) \mathbb{P}^*(W_{T-\delta}^* \in \mathrm{d}y)
\end{aligned}$$

where

$$
\begin{aligned}
f(y) &= \frac{\ln\frac{K}{S_0} - \sigma y - \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{\delta}} \\
g(y) &= \frac{\ln\frac{K}{S_0} - \sigma y - \left(r - \frac{1}{2}\sigma^2\right)T - \sigma^2\delta}{\sigma\sqrt{\delta}} \\
s(y) &= S_0 \exp\left[\sigma y + \left(r - \frac{1}{2}\sigma^2\right)(T-\delta)\right] \\
\mathbb{P}^*(W^*_{T-\delta} \in \mathrm{d}y) &= \frac{\exp\left[-\frac{y^2}{2(T-\delta)}\right]}{\sqrt{2\pi(T-\delta)}}\mathrm{d}y.
\end{aligned}
$$

So, we can re-express the seller's price of $\varepsilon$ puts as

$$
p(\varepsilon) = \varepsilon a + \frac{\gamma}{2}\mathrm{e}^{-rT}\varepsilon^2[\mathbb{E}^* M_T^2 - \mathbb{E}^* M_{T-\delta}^2 - \delta\sigma^2\mathrm{e}^{2rT}S_0^2\Phi^2(-d_1^0)].
$$

Repeating this analysis for the buyer gives us

$$
\begin{aligned}
p^{\mathrm{b}}(\varepsilon) &= -\frac{1}{\gamma\mathrm{e}^{rT}}\ln\left[1 - \varepsilon a\gamma\mathrm{e}^{rT} + \frac{1}{2}\gamma^2\varepsilon^2\left(a^2\mathrm{e}^{2rT} + A\right)\right] \\
&\simeq \varepsilon a - \frac{\gamma}{2}\mathrm{e}^{-rT}\varepsilon^2 A
\end{aligned}
$$

and since $A$ is positive we can see immediately that

$$
p^{\mathrm{b}}(\varepsilon) \leq \varepsilon a \leq p(\varepsilon)
$$

with strict inequalities in the time-lag case. However, when there is no time-lag, i.e. $\delta = 0$, we have

$$
p^{\mathrm{b}}(\varepsilon) = \varepsilon a = p(\varepsilon),
$$

with both the bid and ask unit prices equal to the Black-Scholes price.

# Chapter 6

# Results

In this chapter I present and compare the results obtained from the continuous-time and discrete-time methods described in Chapter 5. We looked at three put options, altering the riskless rate of interest; the volatility and starting price of the risky asset; and the strike price of the option. These parameters are given in Table 6.1. The table also shows the Black-Scholes price of each option with which we compare the continuous-time lagged price.

Table 6.1: Parameters

| Example | 1 | 2 | 3 |
|---|---|---|---|
| $w$ | 0 | 0 | 0 |
| $\sigma^2$ | 0.09 | 0.08 | 0.08 |
| $r$ | $\ln(1.1)$ | $\ln(1.05)$ | $\ln(1.05)$ |
| $S_0$ | 0.95 | 0.95 | 1.0 |
| $K$ | 1.0 | 1.0 | 0.95 |
| $\mu$ | $\ln(1.15)$ | $\ln(1.15)$ | $\ln(1.15)$ |
| $\gamma$ | 2 | 2 | 2 |
| $M^*$ | 300 | 300 | 300 |
| $T$ | 1 | 1 | 1 |
| Black-Scholes price of 1 put | 0.09156189 | 0.10816829 | 0.06607381 |

* $M+1$ is the number of points over which maximisation takes place in the discrete-time case.

We calculated the seller's price of 0.1, 0.01 and 0.005 puts (scaled to unit price). The continuous-time method relies on $\varepsilon$ being small, but we found that $\varepsilon = 0.005$ was sufficient to give smooth graphs and a close comparison between the continuous- and discrete-time results. Table 6.2 shows the discrete-time results for Example 1. In the no-lag case (shown in Column 4) the unit price of $\varepsilon$ puts is almost constant as $\varepsilon$ decreases, reflecting the fact that in the no-lag discrete case, the seller's price is virtually equal to the Black-Scholes price which *is* constant as $\varepsilon$ changes. In the lag case, however, the price decreases as $\varepsilon$ decreases. The seller of 0.005 puts is prepared to accept a lower price than the seller of 0.1 puts. The effect of the lag is shown in the final column as the difference between the price in the lagged economy and the price in the no-lag economy. The seller's price in the lagged economy is higher than in the no-lag economy because his trading is not as efficient and he will need more initial wealth to replicate the put's payoff. The effect is increasing and approximately linear in $\varepsilon$.

Table 6.3 contains the continuous-time results for Example 1. Column 4 shows the lagged economy seller's unit price of $\varepsilon$ put options. Column 5 shows the difference between this price and the Black-Scholes price shown in Table 6.1. For each $\varepsilon$ and $\delta$ the effect is of the same order as the corresponding effect in the discrete-time case and as $\varepsilon$ decreases the effects get closer to the discrete-time effects, agreeing to 1 or 2 significant figures. Considering that both methods use approximations we would not expect to have more agreement than this.

Table 6.2: Example 1: Discrete time

| $\varepsilon$ | $N$ | $\delta = 1/N$ | Price with no lag | Price with lag | Change in price |
|---|---|---|---|---|---|
| 0.1 | 10 | 0.1000 | 9.4904261756269e-2 | 9.5005353533425e-2 | 1.01092e-4 |
| | 12 | 0.0833 | 9.4550956181924e-2 | 9.4636043749777e-2 | 8.50876e-5 |
| | 14 | 0.0714 | 9.4265137253304e-2 | 9.4338794394285e-2 | 7.36571e-5 |
| | 16 | 0.0625 | 9.4027829068302e-2 | 9.4092901253587e-2 | 6.50722e-5 |
| | 20 | 0.0500 | 9.3653455160308e-2 | 9.3706466064906e-2 | 5.30109e-5 |
| | 32 | 0.03125 | 9.2956707828342e-2 | 9.2991418198604e-2 | 3.47104e-5 |
| | 40 | 0.0250 | 9.2668121165887e-2 | 9.2696621777573e-2 | 2.85006e-5 |
| | 50 | 0.0200 | 9.2405016155586e-2 | 9.2428478159222e-2 | 2.34620e-5 |
| | 80 | 0.0125 | 9.1927114048731e-2 | 9.1942823760680e-2 | 1.57097e-5 |
| 0.01 | 10 | 0.1000 | 9.4904261756280e-2 | 9.4914385004189e-2 | 1.01232e-5 |
| | 12 | 0.0833 | 9.4550956181970e-2 | 9.4559475618887e-2 | 8.51944e-6 |
| | 14 | 0.0714 | 9.4265137253328e-2 | 9.4272511406413e-2 | 7.37415e-6 |
| | 16 | 0.0625 | 9.4027829068324e-2 | 9.4034343164112e-2 | 6.51410e-6 |
| | 20 | 0.0500 | 9.3653455160308e-2 | 9.3658761135943e-2 | 5.30598e-6 |
| | 32 | 0.03125 | 9.2956707828327e-2 | 9.2960181264545e-2 | 3.47344e-6 |
| | 40 | 0.0250 | 9.2668121165851e-2 | 9.2670972953953e-2 | 2.85179e-6 |
| | 50 | 0.0200 | 9.2405016155642e-2 | 9.2407363609731e-2 | 2.34745e-6 |
| | 80 | 0.0125 | 9.1927114048538e-2 | 9.1928685681089e-2 | 1.57163e-6 |
| 0.005 | 10 | 0.1000 | 9.4904261756308e-2 | 9.4909323726704e-2 | 5.06198e-6 |
| | 12 | 0.0833 | 9.4550956182000e-2 | 9.4555216166602e-2 | 4.25998e-6 |
| | 14 | 0.0714 | 9.4265137253364e-2 | 9.4268824538328e-2 | 3.68728e-6 |
| | 16 | 0.0625 | 9.4027829068342e-2 | 9.4031086288810e-2 | 3.25722e-6 |
| | 20 | 0.0500 | 9.3653455160310e-2 | 9.3656108273200e-2 | 2.65311e-6 |
| | 32 | 0.03125 | 9.2956707828338e-2 | 9.2958444610466e-2 | 1.73678e-6 |
| | 40 | 0.0250 | 9.2668121165826e-2 | 9.2669547106592e-2 | 1.42594e-6 |
| | 50 | 0.0200 | 9.2405016155618e-2 | 9.2406189917092e-2 | 1.17376e-6 |
| | 80 | 0.0125 | 9.1927114048320e-2 | 9.1927899883636e-2 | 7.85835e-7 |

Table 6.3: Example 1: Continuous time

| $\varepsilon$ | $N$ | $\delta = 1/N$ | Price with lag | Change in price |
|---|---|---|---|---|
| 0.1 | 10 | 0.1000 | 9.1656170567777e-2 | 9.42809e-5 |
| | 12 | 0.0833 | 9.1641765773792e-2 | 7.98761e-5 |
| | 14 | 0.0714 | 9.1631050936184e-2 | 6.91613e-5 |
| | 16 | 0.0625 | 9.1622762031386e-2 | 6.08724e-5 |
| | 20 | 0.0500 | 9.1610752557962e-2 | 4.88629e-5 |
| | 32 | 0.03125 | 9.1591709083999e-2 | 2.98194e-5 |
| | 40 | 0.0250 | 9.1585034332136e-2 | 2.31447e-5 |
| | 50 | 0.0200 | 9.1579553430990e-2 | 1.76638e-5 |
| | 80 | 0.0125 | 9.1571055723562e-2 | 9.16607e-6 |
| 0.01 | 10 | 0.1000 | 9.1572049585047e-2 | 1.01599e-5 |
| | 12 | 0.0833 | 9.1570582724880e-2 | 8.69307e-6 |
| | 14 | 0.0714 | 9.1569491621075e-2 | 7.60197e-6 |
| | 16 | 0.0625 | 9.1568647554449e-2 | 6.75790e-6 |
| | 20 | 0.0500 | 9.1567424621674e-2 | 5.53497e-6 |
| | 32 | 0.03125 | 9.1565485418443e-2 | 3.59577e-6 |
| | 40 | 0.0250 | 9.1564805728159e-2 | 2.91608e-6 |
| | 50 | 0.0200 | 9.1564247608485e-2 | 2.35796e-6 |
| | 80 | 0.0125 | 9.1563382289025e-2 | 1.49264e-6 |
| 0.005 | 10 | 0.1000 | 9.1566990213502e-2 | 5.10056e-6 |
| | 12 | 0.0833 | 9.1566256044231e-2 | 4.36639e-6 |
| | 14 | 0.0714 | 9.1565709942519e-2 | 3.82029e-6 |
| | 16 | 0.0625 | 9.1565287483868e-2 | 3.39783e-6 |
| | 20 | 0.0500 | 9.1564675401251e-2 | 2.78575e-6 |
| | 32 | 0.03125 | 9.1563704822501e-2 | 1.81517e-6 |
| | 40 | 0.0250 | 9.1563364634899e-2 | 1.47498e-6 |
| | 50 | 0.0200 | 9.1563085293844e-2 | 1.19564e-6 |
| | 80 | 0.0125 | 9.1562652198114e-2 | 7.62548e-7 |

The tables for Examples 2 and 3 are given later in the chapter and show similar properties. For Example 1, we also looked at the buyer's prices and the effect of the lag on him. Tables 6.4 and 6.5 respectively show the discrete- and continuous-time results for 0.005 puts. In both cases the effect of the lag is to lower the bid price, since the buyer is less efficient at trading in the lagged case and therefore has less money to risk on the option. In discrete-time the no-lag buyer's price is almost equal to the no-lag seller's price, while the lagged buyer's price is lower than the lagged seller's price. In both cases the effect of the lag on the buyer's price is of the same magnitude as the effect on the seller's price to about 3 significant figures in discrete-time and to 1 or 2 significant figures in continuous-time.

Table 6.4: Example 1: Discrete time bid price (0.005 puts)

| $N$ | $\delta = 1/N$ | Price with no lag | Price with lag | Change in price |
|---|---|---|---|---|
| 10 | 0.1000 | 9.4904261756232e-2 | 9.4899198927758e-2 | 5.06283e-6 |
| 12 | 0.0833 | 9.4550956181833e-2 | 9.4546695539307e-2 | 4.26064e-6 |
| 14 | 0.0714 | 9.4265137253253e-2 | 9.4261449448164e-2 | 3.68781e-6 |
| 16 | 0.0625 | 9.4027829068270e-2 | 9.4024571424790e-2 | 3.25764e-6 |
| 20 | 0.0500 | 9.3653455160297e-2 | 9.3650801748989e-2 | 2.65341e-6 |
| 32 | 0.03125 | 9.2956707828319e-2 | 9.2954970902920e-2 | 1.73693e-6 |
| 40 | 0.0250 | 9.2668121165938e-2 | 9.2666695122759e-2 | 1.42604e-6 |
| 50 | 0.0200 | 9.2405016155507e-2 | 9.2403842320431e-2 | 1.17384e-6 |
| 80 | 0.0125 | 9.1927114049210e-2 | 9.1926328175788e-2 | 7.85873e-7 |

Table 6.5: Example 1: Continuous time bid price (0.005 puts)

| $N$ | $\delta = 1/N$ | Price with lag | Change in price |
|---|---|---|---|
| 10 | 0.1000 | 9.1556747810690e-2 | 5.14184e-6 |
| 12 | 0.0833 | 9.1557483460324e-2 | 4.40619e-6 |
| 14 | 0.0714 | 9.1558030663201e-2 | 3.85899e-6 |
| 16 | 0.0625 | 9.1558453973693e-2 | 3.43568e-6 |
| 20 | 0.0500 | 9.1559067290508e-2 | 2.82236e-6 |
| 32 | 0.03125 | 9.1560039826314e-2 | 1.84982e-6 |
| 40 | 0.0250 | 9.1560380699889e-2 | 1.50895e-6 |
| 50 | 0.0200 | 9.1560660604194e-2 | 1.22905e-6 |
| 80 | 0.0125 | 9.1561094573224e-2 | 7.95077e-7 |

Table 6.6: Example 2: Discrete time

| $\varepsilon$ | $N$ | $\delta = 1/N$ | Price with no lag | Price with lag | Change in price |
|---|---|---|---|---|---|
| 0.1 | 10 | 0.1000 | 0.11187305098480 | 0.11196577049355 | 9.27195e-5 |
| | 12 | 0.0833 | 0.11145886072016 | 0.11153680957977 | 7.79489e-5 |
| | 14 | 0.0714 | 0.11112720106486 | 0.11119464547147 | 6.74444e-5 |
| | 16 | 0.0625 | 0.11085393541461 | 0.11091351428730 | 5.95789e-5 |
| | 20 | 0.0500 | 0.11042638092893 | 0.11047494209759 | 4.85612e-5 |
| | 32 | 0.03125 | 0.10964096411272 | 0.10967287008466 | 3.19060e-5 |
| | 40 | 0.0250 | 0.10931918264485 | 0.10934544805609 | 2.62654e-5 |
| | 50 | 0.0200 | 0.10902746749996 | 0.10904915642206 | 2.16889e-5 |
| | 80 | 0.0125 | 0.10850139065288 | 0.10851602915445 | 1.46385e-5 |
| 0.01 | 10 | 0.1000 | 0.11187305098480 | 0.11188233505112 | 9.28407e-6 |
| | 12 | 0.0833 | 0.11145886072016 | 0.11146666477578 | 7.80406e-6 |
| | 14 | 0.0714 | 0.11112720106486 | 0.11113395274398 | 6.75168e-6 |
| | 16 | 0.0625 | 0.11085393541460 | 0.11085989920603 | 5.96379e-6 |
| | 20 | 0.0500 | 0.11042638092891 | 0.11043124125372 | 4.86032e-6 |
| | 32 | 0.03125 | 0.10964096411272 | 0.10964415679877 | 3.19269e-6 |
| | 40 | 0.0250 | 0.10931918264484 | 0.10932181070281 | 2.62806e-6 |
| | 50 | 0.0200 | 0.10902746750005 | 0.10902963750552 | 2.17001e-6 |
| | 80 | 0.0125 | 0.10850139065285 | 0.10850285510674 | 1.46445e-6 |
| 0.005 | 10 | 0.1000 | 0.11187305098479 | 0.11187769335577 | 4.64238e-6 |
| | 12 | 0.0833 | 0.11145886072016 | 0.11146276300244 | 3.90228e-6 |
| | 14 | 0.0714 | 0.11112720106484 | 0.11113057709958 | 3.37604e-6 |
| | 16 | 0.0625 | 0.11085393541458 | 0.11085691747587 | 2.98206e-6 |
| | 20 | 0.0500 | 0.11042638092886 | 0.11042881121023 | 2.43028e-6 |
| | 32 | 0.03125 | 0.10964096411269 | 0.10964256051421 | 1.59640e-6 |
| | 40 | 0.0250 | 0.10931918264482 | 0.10932049671623 | 1.31407e-6 |
| | 50 | 0.0200 | 0.10902746750016 | 0.10902855253371 | 1.08503e-6 |
| | 80 | 0.0125 | 0.10850139065281 | 0.10850212289628 | 7.32244e-7 |

Table 6.7: Example 2: Continuous time

| $\varepsilon$ | $N$ | $\delta = 1/N$ | Price with lag | Change in price |
|---|---|---|---|---|
| 0.1 | 10 | 0.1000 | 0.10825087905187 | 8.25870e-5 |
| | 12 | 0.0833 | 0.10823774441149 | 6.94523e-5 |
| | 14 | 0.0714 | 0.10822797108759 | 5.96790e-5 |
| | 16 | 0.0625 | 0.10822040459999 | 5.21125e-5 |
| | 20 | 0.0500 | 0.10820943558670 | 4.11435e-5 |
| | 32 | 0.03125 | 0.10819202745139 | 2.37354e-5 |
| | 40 | 0.0250 | 0.10818592181251 | 1.76297e-5 |
| | 50 | 0.0200 | 0.10818090668888 | 1.26146e-5 |
| | 80 | 0.0125 | 0.10817312840728 | 4.83633e-6 |
| 0.01 | 10 | 0.1000 | 0.10817756196760 | 9.26989e-6 |
| | 12 | 0.0833 | 0.10817622135383 | 7.92928e-6 |
| | 14 | 0.0714 | 0.10817522382204 | 6.931744e-6 |
| | 16 | 0.0625 | 0.10817445153633 | 6.15946e-6 |
| | 20 | 0.0500 | 0.10817333196853 | 5.03989e-6 |
| | 32 | 0.03125 | 0.10817155518792 | 3.26311e-6 |
| | 40 | 0.0250 | 0.10817093201066 | 2.63993e-6 |
| | 50 | 0.0200 | 0.10817042013835 | 2.12806e-6 |
| | 80 | 0.0125 | 0.10816962624333 | 1.33417e-6 |
| 0.005 | 10 | 0.1000 | 0.10817295553918 | 4.66346e-6 |
| | 12 | 0.0833 | 0.10817228447047 | 3.99239e-6 |
| | 14 | 0.0714 | 0.10817178513771 | 3.49306e-6 |
| | 16 | 0.0625 | 0.10817139855600 | 3.10648e-6 |
| | 20 | 0.0500 | 0.10817083813591 | 2.54606e-6 |
| | 32 | 0.03125 | 0.10816994873597 | 1.65666e-6 |
| | 40 | 0.0250 | 0.10816963679323 | 1.34472e-6 |
| | 50 | 0.0200 | 0.10816938056622 | 1.08849e-6 |
| | 80 | 0.0125 | 0.10816898316759 | 6.91090e-7 |

Table 6.8: Example 3: Discrete time

| $\varepsilon$ | $N$ | $\delta = 1/N$ | Price with no lag | Price with lag | Change in price |
|---|---|---|---|---|---|
| 0.1 | 10 | 0.1000 | 6.9528692980947e-2 | 6.9620420771858e-2 | 9.17278e-5 |
| | 12 | 0.0833 | 6.9160868768239e-2 | 6.9237181047041e-2 | 7.63123e-5 |
| | 14 | 0.0714 | 6.8862630040862e-2 | 6.8928132145566e-2 | 6.55021e-5 |
| | 16 | 0.0625 | 6.8614595454073e-2 | 6.8672094480109e-2 | 5.74990e-5 |
| | 20 | 0.0500 | 6.8222602570115e-2 | 6.8269035033607e-2 | 4.64325e-5 |
| | 32 | 0.03125 | 6.7491048827856e-2 | 6.7521109975115e-2 | 3.00611e-5 |
| | 40 | 0.0250 | 6.7187360482423e-2 | 6.7211986440786e-2 | 2.46260e-5 |
| | 50 | 0.0200 | 6.6910169759931e-2 | 6.6930430247655e-2 | 2.02605e-5 |
| | 80 | 0.0125 | 6.6405962716031e-2 | 6.6419579145609e-2 | 1.36164e-5 |
| 0.01 | 10 | 0.1000 | 6.9528692980946e-2 | 6.9537878962094e-2 | 9.18598e-6 |
| | 12 | 0.0833 | 6.9160868768258e-2 | 6.9168509646030e-2 | 7.64088e-6 |
| | 14 | 0.0714 | 6.8862630040839e-2 | 6.8869187644371e-2 | 6.55760e-6 |
| | 16 | 0.0625 | 6.8614595454050e-2 | 6.8620351235931e-2 | 5.75578e-6 |
| | 20 | 0.0500 | 6.8222602570087e-2 | 6.8227249841124e-2 | 4.64727e-6 |
| | 32 | 0.03125 | 6.7491048827859e-2 | 6.7494056803069e-2 | 3.00798e-6 |
| | 40 | 0.0250 | 6.7187360482414e-2 | 6.7189824395386e-2 | 2.46391e-6 |
| | 50 | 0.0200 | 6.6910169760027e-2 | 6.6912196758573e-2 | 2.02700e-6 |
| | 80 | 0.0125 | 6.6405962716028e-2 | 6.6407324867935e-2 | 1.36215e-6 |
| 0.005 | 10 | 0.1000 | 6.9528692980952e-2 | 6.9533286331770e-2 | 4.59336e-6 |
| | 12 | 0.0833 | 6.9160868768248e-2 | 6.9164689483734e-2 | 3.82072e-6 |
| | 14 | 0.0714 | 6.8862630040834e-2 | 6.8865909047902e-2 | 3.27900e-6 |
| | 16 | 0.0625 | 6.8614595454010e-2 | 6.8617473511868e-2 | 2.87806e-6 |
| | 20 | 0.0500 | 6.8222602570038e-2 | 6.8224926318422e-2 | 2.32374e-6 |
| | 32 | 0.03125 | 6.7491048827824e-2 | 6.7492552865122e-2 | 1.50404e-6 |
| | 40 | 0.0250 | 6.7187360482390e-2 | 6.7188592474820e-2 | 1.23199e-6 |
| | 50 | 0.0200 | 6.6910169760132e-2 | 6.6911183285548e-2 | 1.01353e-6 |
| | 80 | 0.0125 | 6.6405962716020e-2 | 6.6406643806424e-2 | 6.81090e-7 |

Table 6.9: Example 3: Continuous time

| $\varepsilon$ | $N$ | $\delta = 1/N$ | Price with lag | Change in price |
|---|---|---|---|---|
| 0.1 | 10 | 0.1000 | 6.6159035188228e-2 | 8.52292e-5 |
| | 12 | 0.0833 | 6.6146414583255e-2 | 7.26086e-5 |
| | 14 | 0.0714 | 6.6137048067383e-2 | 6.32421e-5 |
| | 16 | 0.0625 | 6.6129810591741e-2 | 5.60046e-5 |
| | 20 | 0.0500 | 6.6119340496956e-2 | 4.55345e-5 |
| | 32 | 0.03125 | 6.6102778760049e-2 | 2.89728e-5 |
| | 40 | 0.0250 | 6.6096986515757e-2 | 2.31806e-5 |
| | 50 | 0.0200 | 6.6092235613231e-2 | 1.84297e-5 |
| | 80 | 0.0125 | 6.6084880081243e-2 | 1.10741e-5 |
| 0.01 | 10 | 0.1000 | 6.6082627339751e-2 | 8.82138e-6 |
| | 12 | 0.0833 | 6.6081349398932e-2 | 7.54344e-6 |
| | 14 | 0.0714 | 6.6080400963753e-2 | 6.59501e-6 |
| | 16 | 0.0625 | 6.6079668112319e-2 | 5.86216e-6 |
| | 20 | 0.0500 | 6.6078607934672e-2 | 4.80198e-6 |
| | 32 | 0.03125 | 6.6076930936108e-2 | 3.12498e-6 |
| | 40 | 0.0250 | 6.6076344429825e-2 | 2.53847e-6 |
| | 50 | 0.0200 | 6.6075863367387e-2 | 2.05741e-6 |
| | 80 | 0.0125 | 6.6075118568762e-2 | 1.31261e-6 |
| 0.005 | 10 | 0.1000 | 6.6078225009418e-2 | 4.41906e-6 |
| | 12 | 0.0833 | 6.6077585595456e-2 | 3.77964e-6 |
| | 14 | 0.0714 | 6.6077111048720e-2 | 3.30510e-6 |
| | 16 | 0.0625 | 6.6076744368646e-2 | 2.93842e-6 |
| | 20 | 0.0500 | 6.6076213911898e-2 | 2.40796e-6 |
| | 32 | 0.03125 | 6.6075374830632e-2 | 1.56887e-6 |
| | 40 | 0.0250 | 6.6075081373946e-2 | 1.27542e-6 |
| | 50 | 0.0200 | 6.6074840675784e-2 | 1.03472e-6 |
| | 80 | 0.0125 | 6.6074468018012e-2 | 6.62060e-7 |

Figures 6-1 to 6-6 show the relationship between $\delta$, the time-lag, and the change in the seller's price of 0.005 puts for each Example 1 to 3 and for discrete- and continuous-time. Each of the graphs shows an approximately linear relationship, increasing in $\delta$ as would be expected. The solid line on each graph is the least squares best linear fit through the points. On the continuous-time graphs it is easier to see that in fact the relationship curves away from the linear fit and is concave. On closer inspection of the discrete-time graphs this curve can also be seen, although in Example 3 the points are in a convex rather than concave pattern.

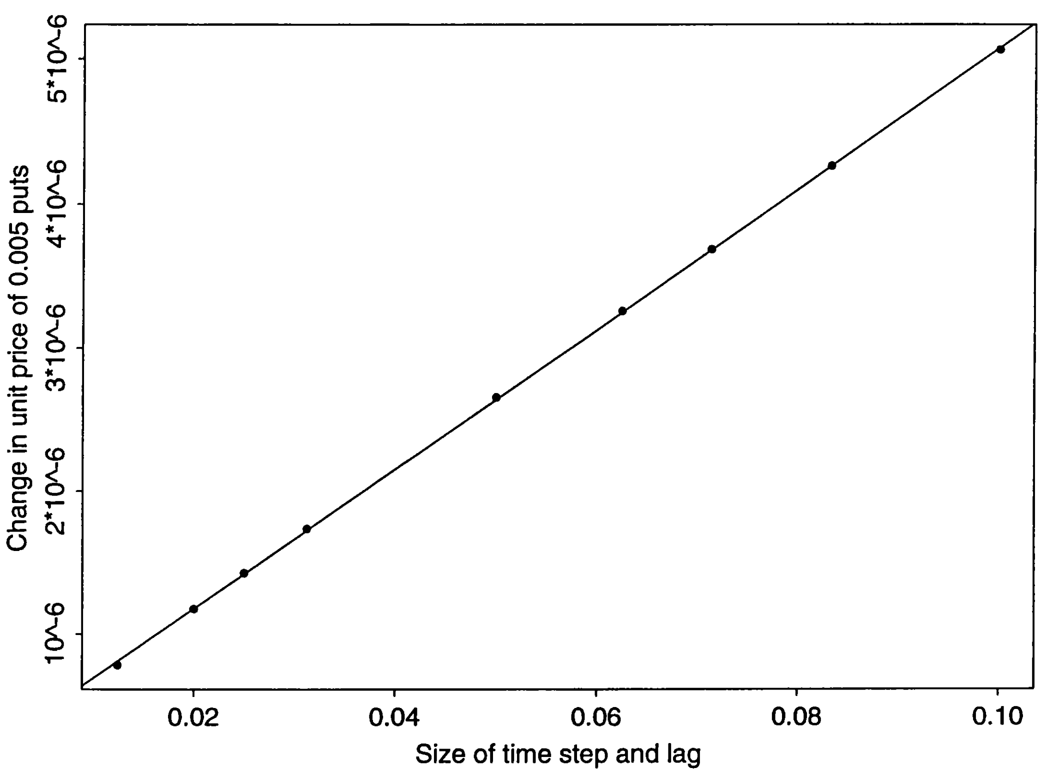


Figure 6-1: Example 1: Discrete time

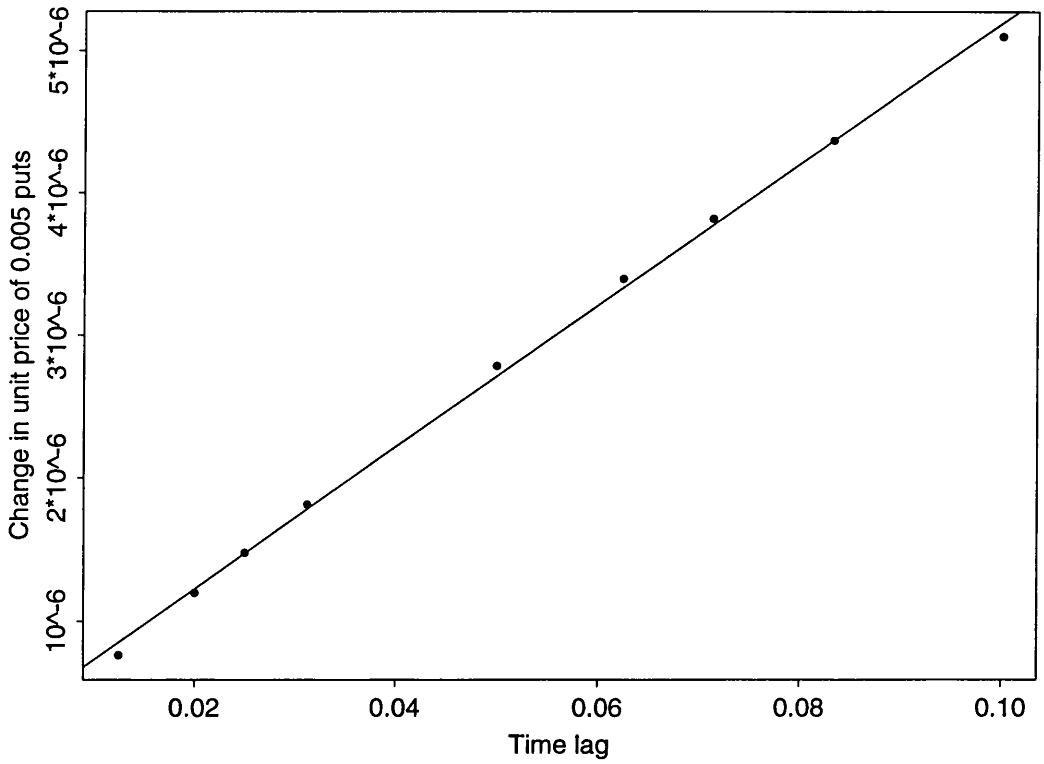


Figure 6-2: Example 1: Continuous time

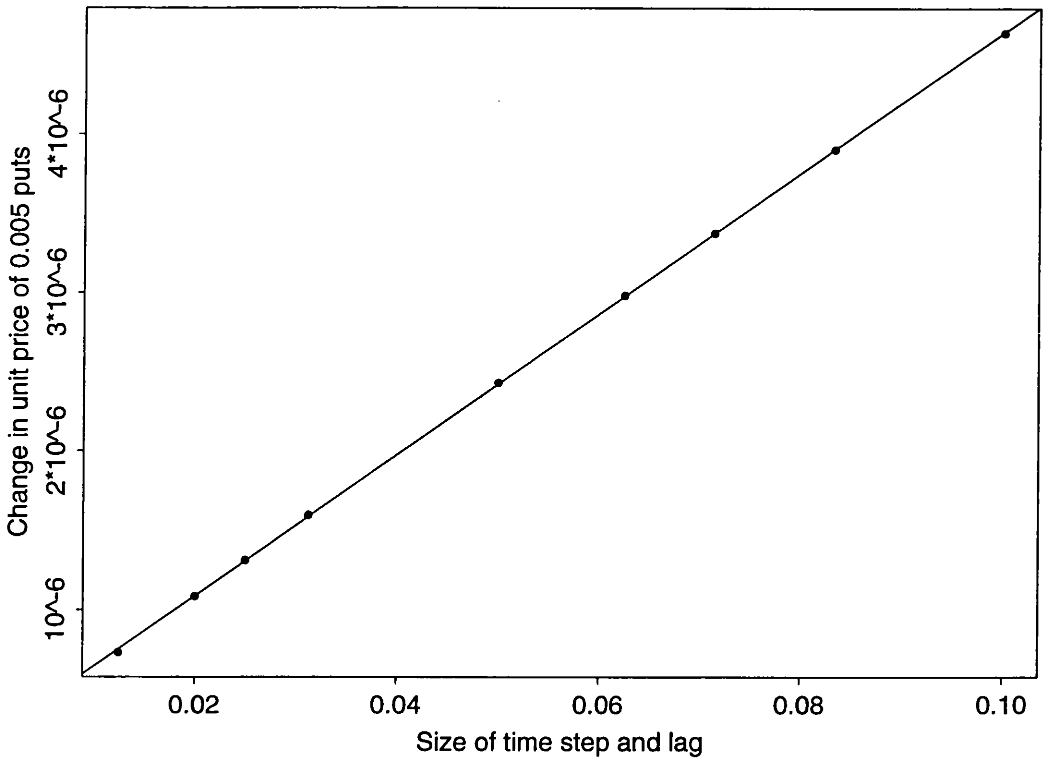


Figure 6-3: Example 2: Discrete time

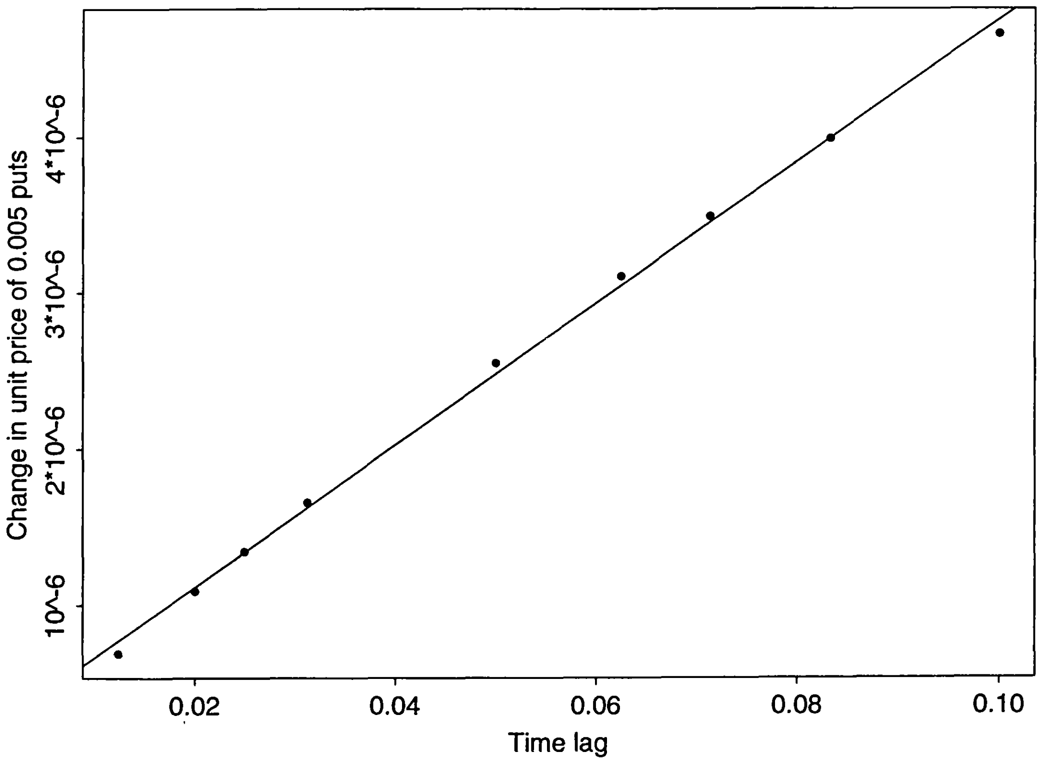


Figure 6-4: Example 2: Continuous time

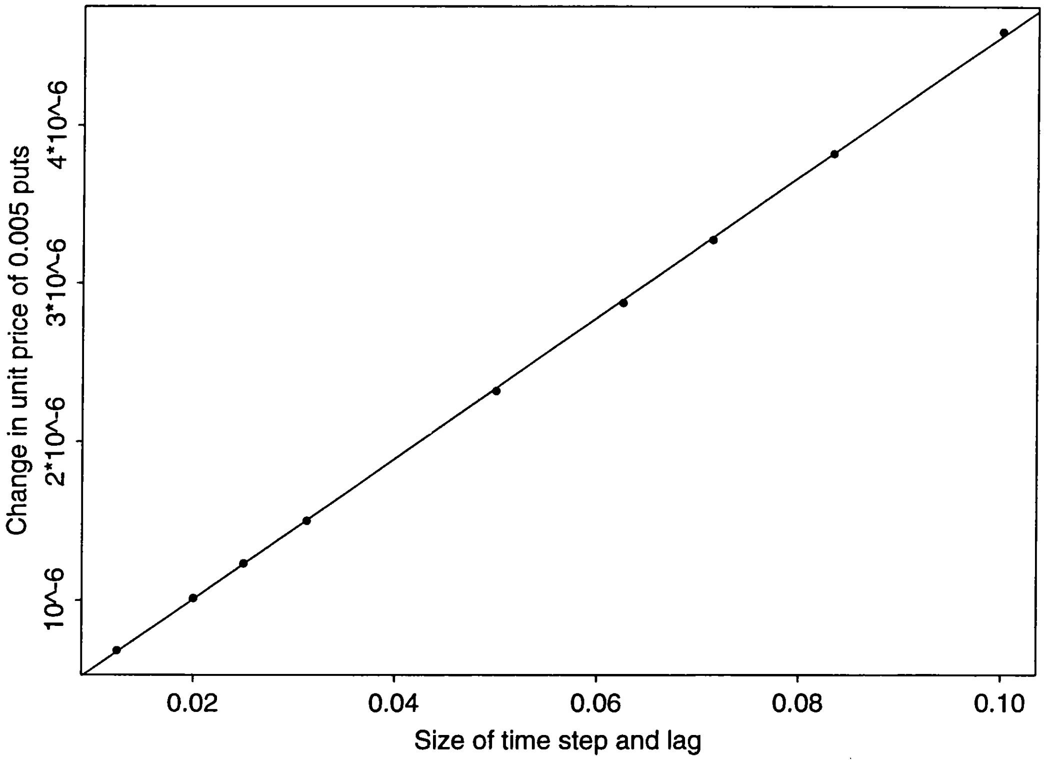


Figure 6-5: Example 3: Discrete time

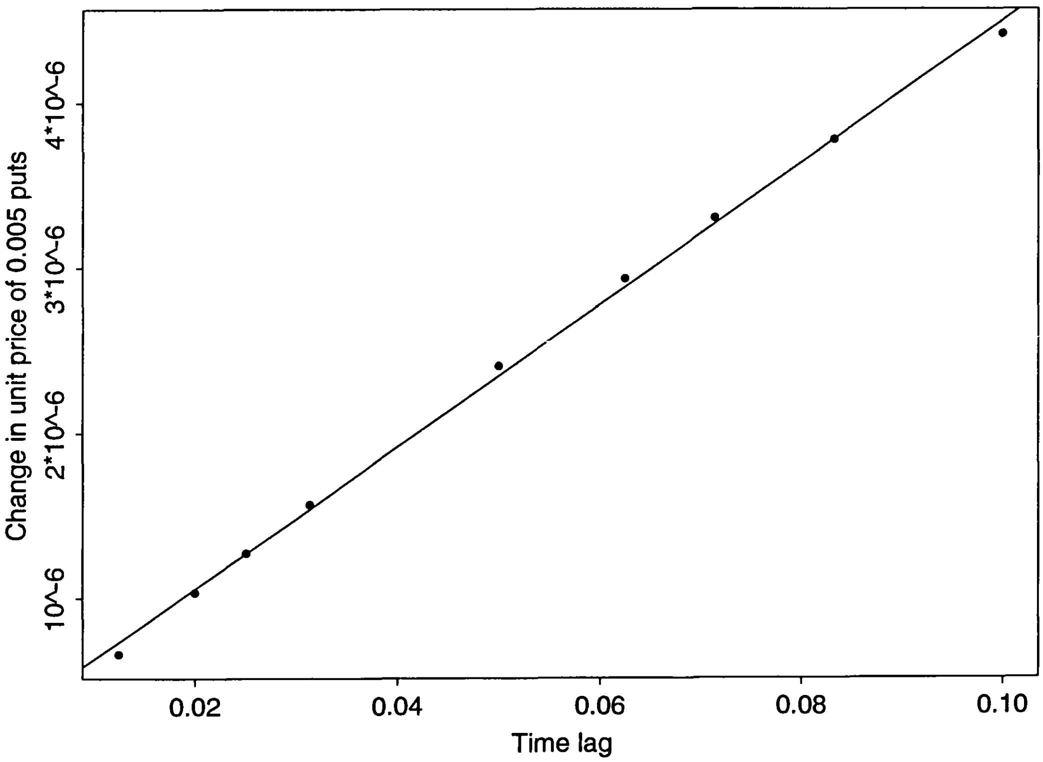


Figure 6-6: Example 3: Continuous time

In summary, Chapters 5 and 6 contain two methods of looking at the effects of a trading time-lag on the seller's and the buyer's prices of $\varepsilon$ put options for investors with exponential utility. Both methods are approximations but give similar results and show an approximately linear relationship between lag and price change with a slightly concave curve in evidence, particularly in the continuous-time results. The effect on the buyer's price is opposite and equal to the effect on the seller's price.

# Chapter 7

# The Power Utility Function

We now look at the investor with constant relative risk aversion, i.e. a power utility function. In this chapter, we look at the effect of the time lag on the maximised expected final utility of such an investor. The effect of a time-lag of length $h$ is to force him to precommit at time $t-h$ to an investment strategy to be implemented at time $t$. This makes the maximisation problem more complicated. While in the no-lag case the optimal strategy can be calculated analytically, when there is a lag each decision must be made separately and depends on the amount committed at the previous time step. With the addition of a trading lag, there is no closed-form solution, so any solution to this problem can only be an approximation. We look only at the discrete-problem here, using two different methods for dealing with this. Firstly, we calculate the value function numerically for four sets of parameter values. The optimal portfolio selection cannot be found precisely but is instead estimated by interpolation between discrete points. Secondly, we approximate the value function by the exponential of a quadratic function in the proportion of wealth invested in the risky asset and, assuming that $h$ is small, ignore terms of order $h^3$ and above in order to obtain an analytic formula for the initial value function as the exponential of a quadratic in $h$. We start by setting up the basic problem common to both methods.

## 7.1 The Set-up

We look at the problem of a single investor with a power utility function given by

$$U(x) = \frac{x^{1-R}}{1-R}$$

defined $\forall\ x > 0$. The constant of relative risk aversion, $R$, is positive, and different from 1. The case $R = 1$ corresponds to logarithmic utility, and could be treated by methods similar to those detailed below.

With initial wealth, $w_0$, the investor aims to maximise the expected utility gained from his final wealth $w_T$, at some fixed finite time $T$ in the future. He achieves this by investing in a risky asset with price process given by exponential Brownian motion

$$S_t = S_0 \exp\left[\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right) t\right]$$

where $\mu, \sigma$ and $S_0$ are constants and $W_t$ is Brownian motion.

There is also a bank account with fixed rate $r$ in which the investor places his remaining wealth. The risky asset can be sold short and money can be borrowed from the bank. However, because of the shape of the utility function close to zero, the investor must always ensure that there is zero probability of his wealth going non-positive at any point. Frictionless markets are assumed with the exception of a small trading time-lag.

We approximate the risky asset price with the discrete-time binomial tree described in Chapter 4. The investor can rebalance his portfolio only at the discrete time steps and the time lag is assumed to be of the same length as one time step, $h$, so that if the investor wishes to change his portfolio at time step $n$, he must make the decision at time step $n - 1$. We now have three discrete-time processes running from time step 0 to time step $N$:

$s_n$ price of the risky asset at time $nh$ ;

$w_n$ wealth at time $nh$ ; and

$\theta_n$ units of the risky asset held throughout the time interval $(nh, nh+h]$.

We also let $x_n = \theta_n s_n$ be the amount of money invested in the risky asset at time $nh$.

We now wish to calculate the value function given by

$$V_n(w,x) \equiv \max \mathbb{E}[U(w_N)|w_n = w, x_n = x]$$

which can also be expressed as

$$\begin{aligned} V_n(w,x) &= \max_{\theta_{n+1}} \{pV_{n+1}(\rho w + x(a-\rho), \theta_{n+1}s_n a) + \\ &\qquad (1-p)V_{n+1}(\rho w + x(1/a-\rho), \theta_{n+1}s_n/a)\} \\ &= \max_{\xi} \{pV_{n+1}(\rho w + x(a-\rho), \xi a) + \\ &\qquad (1-p)V_{n+1}(\rho w + x(1/a-\rho), \xi/a\}\,. \end{aligned}$$

One step before maturity we know that

$$\begin{aligned} V_{N-1}(w,x) &= \mathbb{E}[U(w_N)|w_{N-1} = w, x_{N-1} = x] \\ &= pU(\rho w + x(a-\rho)) + (1-p)U(\rho w + x(1/a-\rho)). \end{aligned}$$

It is easy to show that for any $n$ and $\lambda$

$$V_n(\lambda w, \lambda x) = \lambda^{1-R} V_n(w, x).$$

Therefore, defining $g_n(.) \equiv V_n(1, .)\ \forall\, n$, and substituting $t = x/w$ and $\eta = \xi/w$, so that $t$ is now the proportion of wealth invested in the share at time step $n$, the problem becomes

$$\begin{aligned} g_n(t) \quad = \quad & \max_{\eta} \left\{ p(\rho + t(a-\rho))^{1-R} g_{n+1}\left(\frac{\eta a}{\rho + t(a-\rho)}\right) \right. \\ & \left. +(1-p)(\rho + t(1/a - \rho))^{1-R} g_{n+1}\left(\frac{\eta/a}{\rho + t(1/a-\rho)}\right) \right\} \end{aligned} \tag{7.1}$$

with

$$g_{N-1}(t) = \frac{p(\rho + t(a-\rho))^{1-R} + (1-p)(\rho + t(1/a-\rho))^{1-R}}{1-R}. \tag{7.2}$$

Finally, the maximised expected final utility is given by

$$\begin{aligned} V_0(w_0, x^*) \quad & = \quad w_0^{1-R} g_0\left(\frac{x^*}{w_0}\right) \\ & = \quad w_0^{1-R} g_0(t^*) \end{aligned} \tag{7.3}$$

where $t^*$ is simply the value of $t$ which maximises $g_0(t)$.

## 7.2 The Numerical Method

The problem defined in equations (7.1) and (7.2) can be solved numerically for given parameter values. The calculations are easy to perform but two points should be noted. Firstly, for any $n$, $g_n(t)$ is only defined when both $\rho + t(a-\rho)$ and $\rho + t(1/a - \rho)$ are positive, i.e. when

$$t \in \left(\frac{-\rho}{a-\rho}, \frac{\rho}{\rho - 1/a}\right). \tag{7.4}$$

Also, since the maximising value of $\eta$ in equation (7.1) is dependent on $t$, it is not possible to calculate the function $g_n$ explicitly as a function of $t$. Instead we calculate the value of $g_n$ at a number of equally spaced points within the range in equation (7.4) above. Tests on the no-lag version of this problem (for which a closed-form solution to the discrete-time case is available) showed that 1500 of these points are required to give accuracy to 9 decimal places. However, using so many points makes the algorithm in the lagged case slow to run and so, making the assumption that for a small time-lag and discrete time-step the proportion of wealth invested in the risky asset will be close to the Merton proportion (the optimal proportion of wealth invested in the risky asset for the continuous-time no-lag problem[1]), we restrict our search range to one tenth of the range given above and centre it around the Merton proportion. We can therefore use just 150 points and gain sufficient accuracy.

To obtain the new algorithm let the new search interval have length

$$L = \left[\frac{\rho}{\rho - 1/a} + \frac{\rho}{a-\rho}\right] / 10$$

and start at

$$l = \pi - \frac{L}{2}$$

where $\pi$ is the Merton proportion given by

$$\pi = \frac{\mu - r}{\sigma^2 R}.$$

[1]Derived in Merton (1969).

Finally, let the gridpoints over which the search will be carried out be a distance

$$\varepsilon = \frac{L}{M}$$

apart, where $M+1$ is the number of gridpoints used, equal to 150 in our calculations.

Now we can construct a new recursive function $\tilde{g}$ defined by

$$\begin{aligned}\tilde{g}_n[i] \quad = \quad & \max_{\eta} \left\{ p(\rho + t[i](a-\rho))^{1-R} \tilde{g}_{n+1} \left[ i \left( \frac{\eta a}{\rho + t[i](a-\rho)} \right) \right] \right. \\ & \left. +(1-p)(\rho + t[i](1/a - \rho))^{1-R} \tilde{g}_{n+1} \left[ i \left( \frac{\eta / a}{\rho + t[i](1/a-\rho)} \right) \right] \right\}\end{aligned} \tag{7.5}$$

for $i = 0, ..., M$; $t[i] = l + i\varepsilon$; $i(t) = (t-l)/\varepsilon$; and by interpolation[2] for $i \notin \mathbb{Z}$. The final step is given by

$$\tilde{g}_{N-1}[i] = \frac{p(\rho + t[i](a-\rho))^{1-R} + (1-p)(\rho + t[i](1/a - \rho))^{1-R}}{1-R}.$$

Finally, once we have obtained values for $\tilde{g}_0[i]$ for $i = 1, ..., M-1$, we find $i^*$, the maximising value of $i$, and then perform a cubic interpolation to approximate the maximum more closely. Figure 7-1 is a plot of $\tilde{g}_0[i]$ (for proportion of wealth invested in the risky asset ranging over the values $\pi - L/2$ to $\pi + L/2$) and shows that the maximum is in the middle of the range of values for $i$. It is clear from this that a cubic interpolation will give a reasonable approximation of the maximum. We call this approximation $g^*$.

[2]For $t[i] \in (l + 2\varepsilon, l + L - 2\varepsilon)$ we use cubic interpolation over the four nearest grid points, and for $t$ in either of the end intervals we use quadratic interpolation over the end three points.

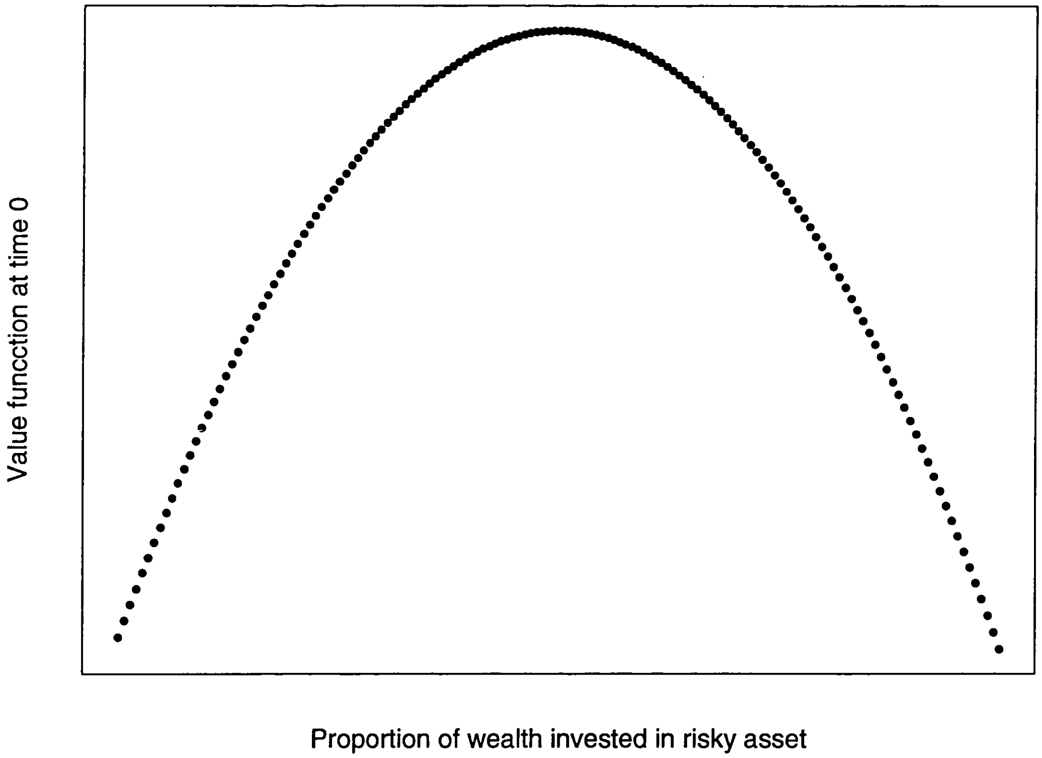


Figure 7-1: Value function with varying investment in risky asset

In Table 7.2 we show the results obtained from this method used on four examples. In the economy without a trading lag there is a closed-form solution obtained from

$$V_n(w) = \max\mathbb{E}[U(w_N)|w_n = w].$$

In this case

$$V_N(w) = U(w)$$

and

$$V_n(w) = \max_x \left\{ pV_{n+1}(\rho w + x(a-\rho)) + (1-p)V_{n+1}(\rho w + x(1/a-\rho)) \right\}.$$

As in the case of the economy *with* a lag, we can show that

$$V_n(\lambda w) = \lambda^{1-R} V_n(w)$$

and if we let $g_n \equiv V_n(1)\ \forall n$,

$$g_N = U(1) = \frac{1}{1-R}$$

and

$$\begin{aligned} g_n &= \max_x \left\{ \left[ p(\rho + x(a-\rho))^{1-R} + (1-p)(\rho + x(1/a-\rho))^{1-R} \right] .g_{n+1} \right\} \\ &\equiv g_{n+1}\xi, \text{ say.} \end{aligned}$$

Therefore $g_0 = g_N \xi^N$.

We also use the corresponding continuous-time results for a market with no time-lag, based on Merton (1969) with consumption omitted. Merton's solution for the current (time 0) value of the maximum expected utility at time $T$, given initial wealth $w_0$ is given by

$$I[w_0, 0] = \frac{w_0^{1-R}}{1-R} \exp\left[ (1-R)\left( r + \frac{(\mu - r)^2}{2R\sigma^2} \right) T \right].$$

Figures 7-2 to 7-5 show plots of $\ln[(1-R)g^*]$ over varying $h$, for each of the four examples. The lines on the graphs are quadratic fits with the least squared errors from the plots and which pass through the Merton solution at $h = 0$.

## 7.3 Asymptotic approximation

We now go back to equations (7.1) and (7.2) and make two simplifying assumptions. Firstly, we assume that $h$ is very small and that terms of order $h^3$ and higher are insignificant. Secondly, for each $n$, we approximate the value function $g_n$ by

$$\begin{aligned} g_n(t) &\approx \frac{1}{1-R}\exp\left[\alpha_n - \gamma_n(t-\beta_n)^2\right] \\ &\equiv \frac{1}{1-R}\exp[Q_n(t)]. \end{aligned} \tag{7.6}$$

We used Maple to perform all the calculations of this section. By taking the log of $(1-R)$ times equation (7.2), expanding in $h$ and then in $t$ about the Merton proportion $\pi$ up to $t^2$ and removing all terms of order $h^4$ and above we obtain an approximation for the quadratic $Q_{N-1}(t)$ and therefore $\alpha_{N-1}$, $\beta_{N-1}$ and $\gamma_{N-1}$:

$$\begin{aligned} \alpha_{N-1} &= \left[r(1-R) + \frac{1}{2}\sigma^2\pi^2R(1-R)\right]h + \frac{1}{12}\sigma^2\pi^2R(R-1)(\pi^2\sigma^2(3R^2+R+1) \\ &\quad +2\pi\sigma^2(2R-1) + 4\pi r(R+1) + 3\sigma^2)h^2 \\ &\quad + \left(7\pi^4\sigma^6R^3 + \pi^2\sigma^6R + 6\pi^3\sigma^6R^2 + 24\pi^5\sigma^6R^3\right. \\ &\quad \left. +12\pi^4\sigma^6R^2 + 12\pi^6\sigma^6R^3\right)h^3] \\ \beta_{N-1} &= \pi - \frac{1}{6}\pi\left(3\sigma^2 + 6R\pi r + 6R\pi\sigma^2 - 3\pi\sigma^2 + 6\pi r + 4\pi^2\sigma^2R^2 + 2\pi^2\sigma^2\right)h \\ \gamma_{N-1} &= \frac{1}{2}R(1-R)\sigma^2h - \frac{1}{4}R\sigma^2(R-1)(2\pi^2\sigma^2(1+R^2) \\ &\quad +2\pi\sigma^2(R-1) + \sigma^2 + 4\pi r(1+R))h^2 \end{aligned} \tag{7.7}$$

Next, taking equation (7.1), we substitute for $g_{n+1}$ using equation (7.6) and again expand in $h$ to order $h^3$. Again, taking the log of $(1-R)$ times this and expanding in $t$ up to order $t^2$ gives us an approximation for $Q_n(t)$ and we have expressions for $\alpha_n$, $\beta_n$ and $\gamma_n$ in terms of $\alpha_{n+1}$, $\beta_{n+1}$ and $\gamma_{n+1}$. These expressions are too long to write down here, but we can first note that $\beta_{n+1}$ and $\gamma_{n+1}$ only appear in the form $\gamma_{n+1}\beta_{n+1}^2$ and then if we look at the first two terms of $\alpha_n$

$$\alpha_{n+1} - \frac{1}{2}(R-1)\frac{\sigma^2\pi^2 R^2(1-R) + 2rR(1-R) - \gamma_{n+1}\beta_{n+1}^2(2\sigma^2 R - 4\sigma^2\pi R - 4r)}{(R(1-R) + 2\gamma_{n+1}\beta_{n+1}^2)}h \tag{7.8}$$

it is easy to see that, to ensure accuracy to order $h^3$, we require only $\gamma_{n+1}\beta_{n+1}^2$ up to at most order $h^2$. The expression for $\gamma_{n+1}$ is of order $h$ and therefore we need only the constant term in $\beta_{n+1}$ and $\gamma_{n+1}$ to order $h^2$. The expressions for $\beta_{n+1}$ and $\gamma_{n+1}$ to the required order are

$$\begin{aligned}
\beta_{n+1} &= \frac{\pi R(1-R) + 2\gamma_{n+2}\beta_{n+2}^2}{R(1-R) + 2\gamma_{n+2}\beta_{n+2}^2} \\
\gamma_{n+1} &= \frac{1}{2}\sigma^2(R(1-R) + 2\gamma_{n+2}\beta_{n+2}^2)h + \frac{1}{4}h^2(-2\pi\sigma^2 R + R\sigma^2 - 2\pi^2\sigma^2 R^2 + 4\pi Rr \\
&\quad -\sigma^2 R^2 + 2\pi^2\sigma^2 R - 2\pi^2\sigma^2 R^4 + 4\sigma^2\pi R^2 - 4\pi R^3 r - 2\sigma^2\pi R^3 + 2\pi^2\sigma^2 R^3 \\
&\quad -\beta_{n+2}^2\gamma_{n+2}(16r - 10\sigma^2 + 24\pi r + 12\sigma^2\pi R - 8\sigma^2 R + 12\sigma^2\pi \\
&\quad -4\sigma^2 R^2 - 16Rr + 8\sigma^2\pi R^2 + 24\pi Rr)\sigma^2
\end{aligned} \tag{7.9}$$

But since $\gamma_{n+2}$ is of order $h$, we can ignore it in the expression for $\beta_{n+1}$ and therefore have

$$\beta_{n+1} = \pi$$

when using $\beta_{n+1}$ in the expression for $\alpha_n$. This is also the case when substituting for $\beta_{n+2}$ in the expression for $\gamma_{n+1}$ since using terms of order $h$ in $\beta_{n+2}$ would contribute terms of order $h^3$ which are neglected. Therefore for our purposes $\beta_n = \pi \ \forall n$.

We can also show that $\forall\, n < N-1$, $\gamma_n$ does not change. Substituting $\beta_{n+1} = \pi$ into equation (7.9) and using equation (7.7) for $\gamma_{n+1}$ we have an expression for $\gamma_{N-2}$. Next using this expression in equation (7.9) and again using $\beta_{n+1} = \pi$ gives us $\gamma_{N-3}$. Our Maple program shows these two expressions to be identical, and it is clear that from this step backwards, since the substitution is now the same each time, we will have the same expression for $\gamma_n$ for each $n$.

So finally, from equation (7.8),

$$\begin{aligned} \alpha_0 &= \alpha_1 + f(h, \beta_1, \gamma_1) \\ &= \alpha_{N-1} + f(h, \beta_{N-1}, \gamma_{N-1}) + \cdots + f(h, \beta_1, \gamma_1) \end{aligned}$$

for some function $f$. Since $\gamma_n$ and $\beta_n$ do not vary for $n < N-1$ this gives

$$\alpha_0 = \alpha_{N-1} + f(h, \beta_{N-1}, \gamma_{N-1}) + (N-2) f(h, \beta_{N-2}, \gamma_{N-2})$$

which is easily calculated by Maple.

Figure 7-1 shows that the value function $\tilde{g}_0[i]$ obtained from the numerical method of Section 7.2 is maximised by some value of $i$ in the middle of the range $1, ..., M-1$. From this we can infer that $t^*$, the maximising value of $g_0(t)$ lies in the interior of the range given by equation (7.4), and therefore that the approximation to $g_0(t)$ given by

$$\frac{1}{1-R} \exp[Q_0(t)]$$

is also maximised in the interior of this range. Therefore, we must have $\gamma_0 > 0$ and $t^*$ must equal $\beta_0$ giving

$$g_0(t^*) = \frac{1}{1-R} \exp[\alpha_0].$$

## 7.4 Results

We used both the numerical and asymptotic methods on four examples with parameters as given in Table 7.1.

Table 7.1: Parameter Values

| Example | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $T$ | 1 | 1 | 1 | 1 |
| $w_0$ | 1 | 1 | 1 | 1 |
| $\mu$ | ln(1.15) | ln(1.15) | ln(1.15) | ln(1.15) |
| $\sigma$ | 0.3 | 0.3 | 0.3 | 0.3 |
| $r$ | ln(1.1) | ln(1.05) | ln(1.1) | ln(1.05) |
| $R$ | 2 | 2 | 4 | 4 |

Table 7.2 shows the results of the numerical method for the expected maximised final utility for the lagged problem. It also shows the closed form solution to the no lag problem and the Merton solution to the continuous-time no lag problem. The final two columns show the effect of the lag when comparing that solution to the discrete solution and to the Merton solution. The effect is obtained from the equation

$$\left(\frac{u_0}{u_h}\right)^{1/(1-R)} - 1$$

where $u_0$ represents the solution to the problem with no lag and $u_h$ represents the solution to the problem with lag $h$. In other words it is the extra proportional wealth required at time 0 to give an investor operating under a time-lag the same final utility of wealth as an investor operating with no lag. The Merton effect is obtained from a similar equation with the Merton solution replacing the no lag solution.

The effect of the lag is very small. This is due to the utility function. Merton (1969) showed that the optimal strategy for an investor in an economy without a lag is to keep a constant proportion of current wealth invested in the risky asset. In an economy *with* a lag, the only disadvantage to the investor is that he does not know what his wealth will be by the time his trade comes into effect. This will minimise the effect of the lag. For a different utility function, the optimal strategy could depend on the share price as well as current wealth, making the effect more significant.

In Table 7.3 we present once again the numerical method results for the lag problem but now take the log of $(1 - R)$ of this value in order to compare with the quadratic approximation obtained from the asymptotic method. Figures 7-2 to 7-5 show, for each example, plots of the final column of Table 7.3. The line on the figures is a least squared error fitted quadratic which must pass through the Merton solution at $h = 0$ and as close as possible to the numerical method solutions. The relationship looks to be almost linear but closer examination of the graphs shows that it is slightly concave for all four examples.

Table 7.4 shows the coefficients of each of these quadratics. The $h^2$ coefficient is small but not close enough to zero for us to say that this relationship is linear. The residual errors of the fits (Table 7.5) are very small, showing not only a strongly quadratic relationship in the numerical results but also a good connection between these results and the Merton solution through which the quadratic fit passes.

Table 7.6 shows the lagged solution using the asymptotic method with $h = 1/512$. Comparison with the second column of Table 7.2 in the 512 step row shows an accuracy up to at least 8 decimal places.

Finally Table 7.7 shows the coefficients of the quadratic approximation obtained from the asymptotic method. The constant coefficient is identically equal to the Merton solution, as is the corresponding coefficient for the numerical solution, by construction of the quadratic fit. The coefficients of $h$ agree to 8 decimal places for Example 1; 7 for Example 2; and 6 for Examples 3 and 4. The quadratic coefficients are not as close, but are of the same order. When multiplied by $h^2$ for $h$ small (e.g. $1/1024$) the difference between the two methods is no more than the residual from the numerical method fit.

Table 7.2: Numerical method expected utility with and without lag

| N | With lag | Without lag | Effect | Merton effect |
|---|---|---|---|---|
| Example 1: Merton soln: -0.90411478205730 | | | | |
| 32 | -0.90414225123187 | -0.90413452902503 | 8.54099e-6 | 3.03824e-5 |
| 64 | -0.90412859153797 | -0.90412468206268 | 4.32404e-6 | 1.52740e-5 |
| 128 | -0.90412170537672 | -0.90411973872227 | 2.17521e-6 | 7.65757e-6 |
| 256 | -0.90411824829823 | -0.90411726205939 | 1.09083e-6 | 3.83385e-6 |
| 512 | -0.90411651626916 | -0.90411602247625 | 5.46161e-7 | 1.91813e-6 |
| 1024 | -0.90411564938315 | -0.90411540237128 | 2.73208e-7 | 9.59309e-7 |
| Example 2: Merton soln: -0.93073686698172 | | | | |
| 32 | -0.93088237505528 | -0.93086778313485 | 1.56756e-5 | 1.56336e-4 |
| 64 | -0.93080990066567 | -0.93080257098379 | 7.87458e-6 | 7.84687e-5 |
| 128 | -0.93077345347814 | -0.93076978085231 | 3.94579e-6 | 3.93092e-5 |
| 256 | -0.93075517754950 | -0.93075333943327 | 1.97487e-6 | 1.96732e-5 |
| 512 | -0.93074602652045 | -0.93074510709260 | 9.87841e-7 | 9.84117e-6 |
| 1024 | -0.93074144774160 | -0.93074098800926 | 4.93942e-7 | 4.92165e-6 |
| Example 3: Merton soln: -0.24838483254467 | | | | |
| 32 | -0.24839690526023 | -0.24839260914534 | 5.76519e-6 | 1.62014e-5 |
| 64 | -0.24839090891915 | -0.24838873027335 | 2.92370e-6 | 8.15445e-6 |
| 128 | -0.24838788058174 | -0.24838678377683 | 1.47190e-6 | 4.09046e-6 |
| 256 | -0.24838635889728 | -0.24838580875406 | 7.38291e-7 | 2.04837e-6 |
| 512 | -0.24838559617823 | -0.24838532079787 | 3.69561e-7 | 1.02480e-6 |
| 1024 | -0.24838521434907 | -0.24838507670845 | 1.84714e-7 | 5.12383e-7 |
| Example 4: Merton soln: -0.27818593730605 | | | | |
| 32 | -0.27825722773644 | -0.27824248380689 | 1.76629e-5 | 8.54157e-5 |
| 64 | -0.27822176054405 | -0.27821430739493 | 8.92967e-6 | 4.29230e-5 |
| 128 | -0.27820389301224 | -0.27820014670467 | 4.48872e-6 | 2.15148e-5 |
| 256 | -0.27819492591561 | -0.27819304811209 | 2.25000e-6 | 1.07704e-5 |
| 512 | -0.27819043405155 | -0.27818949423807 | 1.12611e-6 | 5.38815e-6 |
| 1024 | -0.27818818604106 | -0.27818771615449 | 5.63033e-7 | 2.69452e-6 |

Table 7.3: Numerical method expected final utility for lag problem

| $\delta = 1/N$ | Max Expected Utility | ln((1-R)(Max Expected Utility)) |
|---|---|---|
| | Example 1 | |
| Merton | -0.90411478205730 | -0.10079895534781 |
| 0.0009765625 | -0.90411564938315 | -0.10079799603883 |
| 0.001953125 | -0.90411651626916 | -0.10079703721725 |
| 0.00390625 | -0.90411824829823 | -0.10079512150464 |
| 0.0078125 | -0.90412170537672 | -0.10079129781024 |
| 0.015625 | -0.90412859153797 | -0.10078368142963 |
| 0.03125 | -0.90414225123187 | -0.10076857341198 |
| | Example 2 | |
| Merton | -0.93073686698172 | -0.071778676474742 |
| 0.0009765625 | -0.93074144774160 | -0.071773754838166 |
| 0.001953125 | -0.93074602652045 | -0.071768835354244 |
| 0.00390625 | -0.93075517754950 | -0.071759003473362 |
| 0.0078125 | -0.93077345347814 | -0.071739368074437 |
| 0.015625 | -0.93080990066567 | -0.071700210883378 |
| 0.03125 | -0.93088237505528 | -0.071622352269909 |
| | Example 3 | |
| Merton | -0.24838483254467 | -0.29416370272820 |
| 0.0009765625 | -0.24838521434907 | -0.29416216558077 |
| 0.001953125 | -0.24838559617823 | -0.29416062833602 |
| 0.00390625 | -0.24838635889728 | -0.29415755763510 |
| 0.0078125 | -0.24838788058174 | -0.29415143137355 |
| 0.015625 | -0.24839090891915 | -0.29413923947859 |
| 0.03125 | -0.24839690526023 | -0.29411509902704 |
| | Example 4 | |
| Merton | -0.27818593730605 | -0.18085326096627 |
| 0.0009765625 | -0.27818818604106 | -0.18084517743069 |
| 0.001953125 | -0.27819043405155 | -0.18083709656486 |
| 0.00390625 | -0.27819492591561 | -0.18082094997157 |
| 0.0078125 | -0.27820389301224 | -0.18078871734876 |
| 0.015625 | -0.27822176054405 | -0.18072449481804 |
| 0.03125 | -0.27825722773644 | -0.18059702480323 |

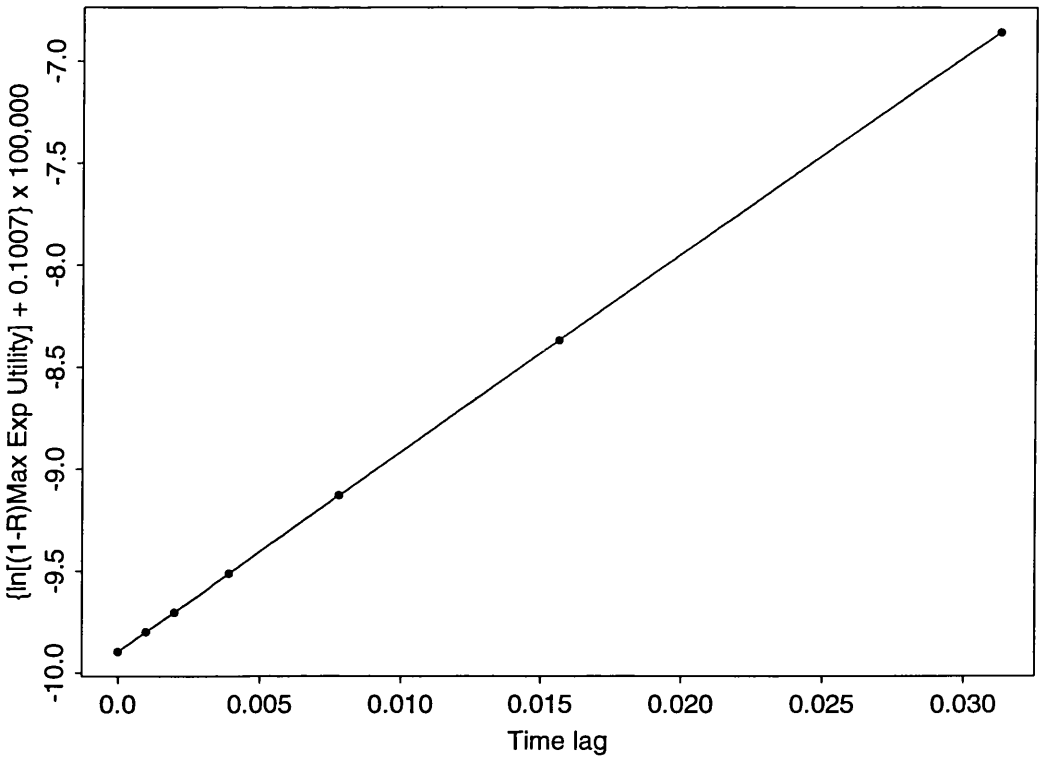


Figure 7-2: Example 1 - numerical results

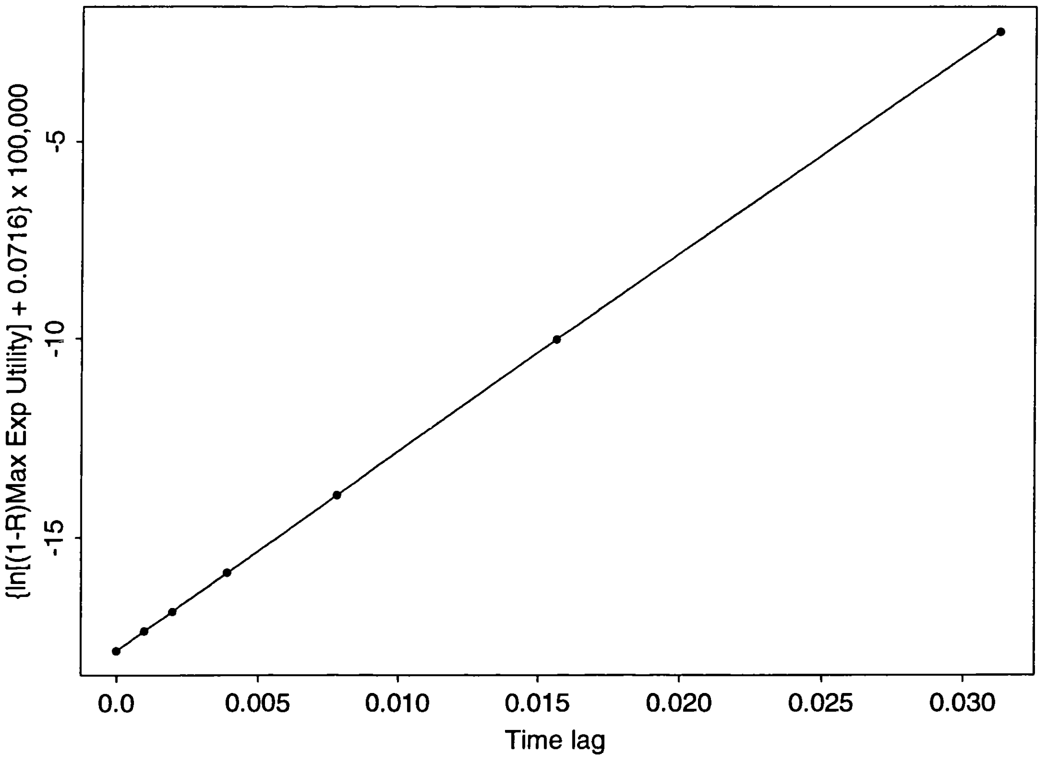


Figure 7-3: Example 2 - numerical results

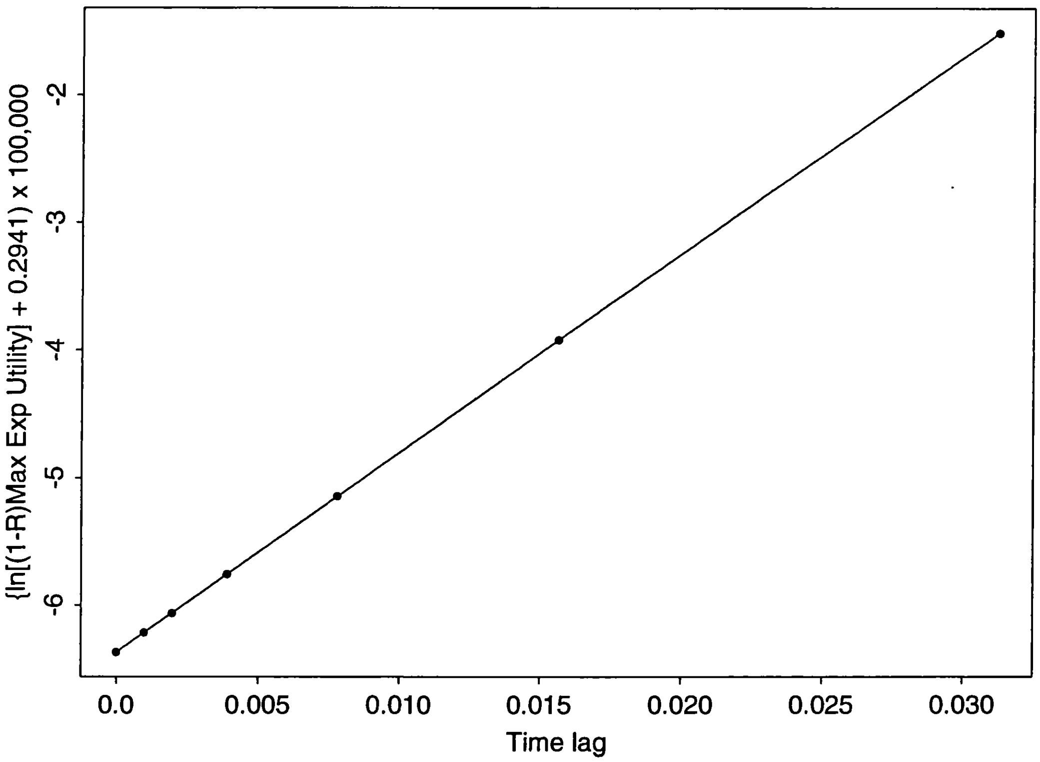


Figure 7-4: Example 3 - numerical results

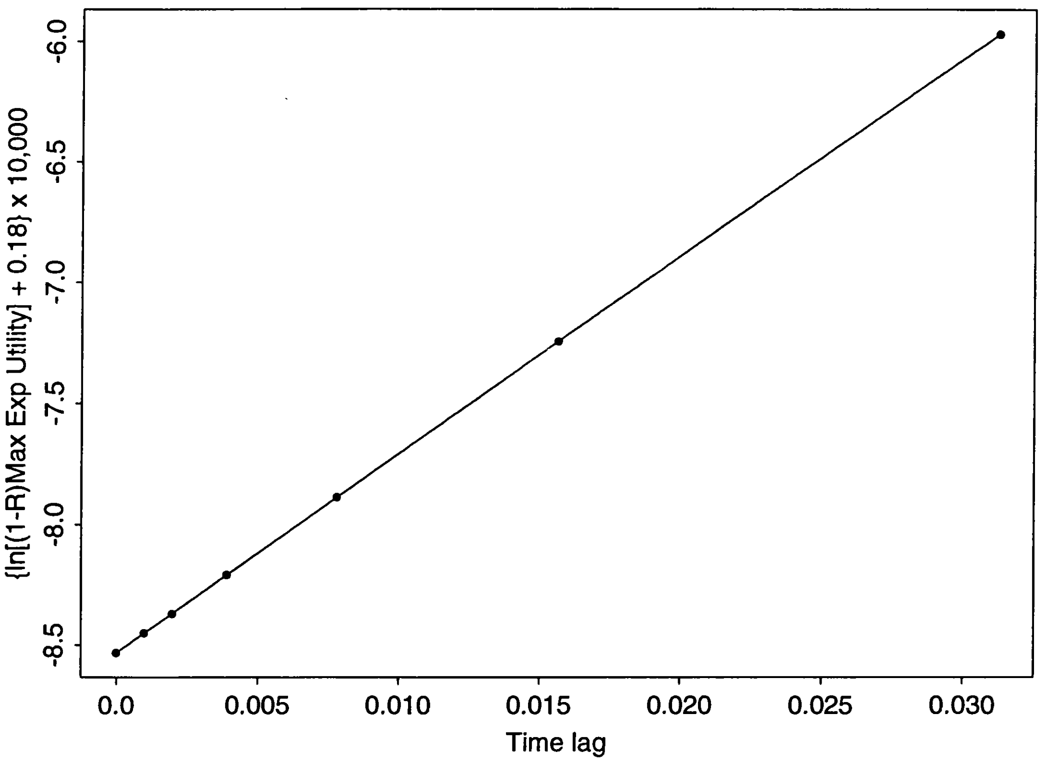


Figure 7-5: Example 4 - numerical results

Table 7.4: Quadratic of least squared error fit to end column of Table 7.3

| Example | Constant | $h$ | $h^2$ |
|---|---|---|---|
| 1 | -0.10079895534781 | 0.0009828157909 | -0.00033900385 |
| 2 | -0.071778676474742 | 0.005041190112 | -0.0012420548 |
| 3 | -0.29416370272820 | 0.001575875023 | -0.00065765049 |
| 4 | -0.18085326096627 | 0.008282313242 | -0.0026478219 |

Table 7.5: Residual errors of quadratic fit to numerical results

| Example | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1024 | 0.2e-9 | 0.22e-9 | 0.12e-8 | 0.21e-8 |
| 512 | 0.2e-9 | 0.22e-9 | 0.10e-8 | 0.19e-8 |
| 256 | 0.2e-9 | 0.20e-9 | 0.6e-9 | 0.14e-8 |
| 128 | 0.1e-9 | 0.9e-10 | 0.1e-9 | 0.3e-9 |
| 64 | -0.2e-9 | -0.23e-9 | -0.7e-9 | -0.15e-8 |
| 32 | 0 | 0.5e-10 | 0.2e-9 | 0.3e-9 |

Table 7.6: Asymptotic method lagged expected utility

| Example | Max Exp Utility |
|---|---|
| 1 | -0.90411651669063927635 |
| 2 | -0.93074602754316439267 |
| 3 | -0.24838559666155399792 |
| 4 | -0.27819043553283254312 |

Table 7.7: Asymptotic quadratic approximation to ln((1-R)(Max Exp Util))

| Example | Constant | $h$ | $h^2$ |
|---|---|---|---|
| 1 | -0.10079895534781 | 0.00098281651791506 | -0.00025343375425765 |
| 2 | -0.07177867647474292 | 0.005041171131728039 | -0.001000880558623706 |
| 3 | -0.29416370272819868 | 0.001576044055769358 | -0.0004909973730133032 |
| 4 | -0.1808532609662624 | 0.008282481294620206 | -0.001833741938090894 |

## 7.5 Summary

In this chapter I explain the two methods used to measure the effect of a time lag on an investor with power utility function. The results of the two methods are shown to agree closely with each other. The effect of the lag is very small. This is due to the utility function. Merton (1969) showed that the optimal strategy for an investor in an economy without a lag is to keep a constant proportion of current wealth invested in the risky asset. In an economy *with* a lag, the only disadvantage to the investor is that he does not know what his wealth will be by the time his trade comes into effect. This will minimise the effect of the lag. For a different utility function, the optimal strategy could depend on the share price as well as current wealth, making the effect more significant.

# Chapter 8

# Appendix

## 8.1 Derivation of the probability density for Brownian motion restricted by an upper and a lower barrier

**Theorem 8.1.1** (Revuz and Yor, p.105 Exercise 3.15.2)

*For every Borel subset $E$ of $[a, b]$, where $a < 0 < b$,*

$$\mathbb{P}[a \leq \hat{W}_t < \bar{W}_t \leq b, W_t \in E] = \int_E k(x)\mathrm{d}x \tag{8.1}$$

*where*

$$k(x) = \frac{1}{\sqrt{2\pi t}} \sum_{k=-\infty}^{\infty} \left\{ \mathrm{e}^{-\frac{1}{2t}(x+2k(b-a))^2} - \mathrm{e}^{-\frac{1}{2t}(x-2b+2k(b-a))^2} \right\}. \tag{8.2}$$

**Proof:** For any $u \in \mathbb{R}$, let $H_u$ be the first hitting time of $u$ by $W$, as defined by equation (1.9), and for any Borel set $F \subset \mathbb{R}$, let

$$s_u F = \{2u - y : y \in F\},$$

the reflection of the set $F$ about the level $u$. Then the following result holds

**Result 8.1.2**

$$\mathbb{P}[H_a < H_b, H_a \leq t, W_t \in E] = \mathbb{P}[H_a < H_b, W_t \in s_a E] \tag{8.3}$$

**Proof:** Using the reflection principle (Rogers and Williams, Volume 1, Section I.13), for any $u \in \mathbb{R}$, the process $W_t^u$ defined by

$$W_t^u = \begin{cases} W_t & t < H_u \\ 2u - W_t & t \geq H_u \end{cases} \tag{8.4}$$

is a Brownian motion. Therefore, in equation (8.3)

$$\begin{aligned} \text{LHS} &= \mathbb{P}[H_a < H_b, H_a \leq t, W_t^a \in E] \\ &= \mathbb{P}[H_a < H_b, H_a \leq t, W_t \in s_a E] \\ &= \text{RHS} \end{aligned}$$

where the final line is obtained from the fact that if $W_t \in s_a E$ then $W_t < a$ and so $H_a \leq t$.

□

We also need

**Result 8.1.3** *If $a < 0 < b$, $F \subset (-\infty, a]$ and $t > 0$:*

$$\mathbb{P}[H_b < H_a, W_t \in F] = \mathbb{P}[W_t \in s_b F] - \mathbb{P}[H_a < H_b, W_t \in s_b F] \tag{8.5}$$

*and similarly for $G \subset [b, \infty)$:*

$$\mathbb{P}[H_a < H_b, W_t \in G] = \mathbb{P}[W_t \in s_a G] - \mathbb{P}[H_b < H_a, W_t \in s_a G] \tag{8.6}$$

**Proof:**

For (8.5):

$$\begin{aligned}
\text{LHS} &= \mathbb{P}[H_b < H_a, W_t^b \in F] \\
&= \mathbb{P}[H_b < H_a, W_t \in s_b F] \\
&= \mathbb{P}[W_t \in s_b F] - \mathbb{P}[H_a < H_b, W_t \in s_b F]
\end{aligned}$$

and (8.6) follows by symmetry.

□

Now returning to the proof of Theorem 8.1.1, applying equations (8.5) and (8.6) repeatedly to the right hand side of equation (8.3) gives:

$$\begin{aligned}
\mathbb{P}[H_a < H_b, W_t \in s_a E] &= \mathbb{P}[W_t \in s_a E] - \mathbb{P}[H_b < H_a, W_t \in s_a E] \\
&= \mathbb{P}[W_t \in s_a E] - \mathbb{P}[W_t \in s_b s_a E] + \mathbb{P}[H_a < H_b, W_t \in s_b s_a E] \\
&= \mathbb{P}[W_t \in s_a E] - \mathbb{P}[W_t \in s_b s_a E] + \mathbb{P}[W_t \in s_a s_b s_a E] \\
&\quad - \mathbb{P}[H_b < H_a, W_t \in s_a s_b s_a E] \\
&\vdots \\
&= \sum_{i=0}^{\infty} \left\{ \mathbb{P}[W_t \in s_a (s_b s_a)^i E] - \mathbb{P}[W_t \in (s_b s_a)^{i+1} E] \right\} \qquad (8.7)
\end{aligned}$$

where $(s_b s_a)^i$ denotes the transformation $(s_b s_a)$ applied $i$ times.

Finally,

$$\begin{aligned}
\mathbb{P}[a \le \hat{W}_t < \bar{W}_t \le b, W_t \in E] &= \mathbb{P}[H_a > t, H_b > t, W_t \in E] \\
&= \mathbb{P}[W_t \in E, H_a > t, H_b > t, H_a < H_b] \\
&\quad + \mathbb{P}[W_t \in E, H_a > t, H_b > t, H_b < H_a] \\
&= \mathbb{P}[W_t \in E] - \mathbb{P}[H_a < H_b, W_t \in E, H_a \le t] \\
&\quad - \mathbb{P}[H_b < H_a, W_t \in E, H_b \le t] \qquad (8.8)
\end{aligned}$$

Now applying equations (8.3) and (8.7) to this gives

$$\begin{aligned}
\mathbb{P}[a \le \hat{W}_t < \bar{W}_t \le b, W_t \in E] &= \mathbb{P}[W_t \in E] - \mathbb{P}[H_a < H_b, W_t \in s_a E] \\
&\quad - \mathbb{P}[H_b < H_a, W_t \in s_b E] \\
&= \mathbb{P}[W_t \in E] - \sum_{i=0}^{\infty} \{\mathbb{P}[W_t \in s_a (s_b s_a)^i E] \\
&\quad - \mathbb{P}[W_t \in (s_b s_a)^{i+1} E]\} \\
&\quad - \sum_{i=0}^{\infty} \{\mathbb{P}[W_t \in s_b (s_a s_b)^i E] - \mathbb{P}[W_t \in (s_a s_b)^{i+1} E]\} \\
&\qquad (8.9)
\end{aligned}$$

and, for example, since

$$(s_b s_a)^i E = \{2i(b-a) + y : y \in E\}$$

we have

$$\mathbb{P}[W_t \in (s_b s_a)^i E] = \int_E \frac{\mathrm{e}^{-\frac{(y+2i(b-a))^2}{2t}}}{\sqrt{2\pi t}} \mathrm{d}y$$

The other densities can be found by symmetry and a little algebra gives us equation (8.1).

$\square$

## 8.2 Extra Proofs for Chapter 5, Section 5.3.4

### 8.2.1 Properties of Semi-groups

Brownian motion is a Markov process with transition semi-group $(P_t)_{t\geq 0}$. Thus for any bounded Borel function $f : \mathbb{R} \to \mathbb{R}$ , and $s, t \geq 0$,

$$\mathbb{E}[f(W_{t+s})|\mathcal{F}_s] = P_t f(W_s)$$

and

$$P_t f(x) := \begin{cases} \int_{-\infty}^{\infty} p_t(x,y) f(y) \mathrm{d}y & (t > 0), \\ f(x) & (t = 0) \end{cases}$$

where $p_t$ is the transition density given by

$$p_t(x,y) := \frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{(x-y)^2}{2t}\right].$$

$(P_t)_{t\geq 0}$ satisfies the semigroup property

$$P_{t+s} = P_t P_s = P_s P_t \;\; (s, t \geq 0).$$

The following two properties of $(P_t)_{t\geq 0}$ are used in Section 5.3:

a) $(\nabla P_t g)(W_s) = (P_t \nabla g)(W_s)$

**Proof**

$$
\begin{aligned}
(\nabla P_t g)(W_s) &= \lim_{h\downarrow 0} \left[\frac{(P_t g)(W_s+h)-(P_t g)(W_s)}{h}\right] \\
&= \lim_{h\downarrow 0} \frac{1}{h}\{\mathbb{E}[g(W_{t+s}+h)-g(W_{t+s})|\mathcal{F}_s]\} \\
&= \mathbb{E}\left[\lim_{h\downarrow 0} \frac{1}{h}\{g(W_{t+s}+h)-g(W_{t+s})\} \mid \mathcal{F}_s\right] \\
&= \mathbb{E}[\nabla g(W_{t+s})|\mathcal{F}_s] \\
&= (P_t \nabla g)(W_s)
\end{aligned}
$$

□

b) $\mathbb{E}[(P_{T-t}\nabla g)(W_t)|\mathcal{F}_{t-\delta}] = (P_{(T-t+\delta)\wedge T}\nabla g)(W_{(t-\delta)^+})$

**Proof**

$$
\begin{aligned}
\mathbb{E}[(P_{T-t}\nabla g)(W_t)|\mathcal{F}_{t-\delta}] &= \mathbb{E}[\mathbb{E}[\nabla g(W_T)|\mathcal{F}_t]|\mathcal{F}_{t-\delta}] \\
&= \mathbb{E}[\nabla g(W_T)|\mathcal{F}_{t-\delta}] \\
&= \begin{cases} (P_{T-t+\delta}\nabla g)(W_{t-\delta}) & \text{if } t > \delta \\ \mathbb{E}[\nabla g(W_T)] & \text{if } 0 \le t < \delta \end{cases} \\
&= (P_{(T-t+\delta)\wedge T}\nabla g)(W_{(t-\delta)^+})
\end{aligned}
$$

□

### 8.2.2 Calculations used in the formula for $\mathbb{E}^* M_T^2$

$$
\begin{aligned}
\mathbb{P}^*(S_T < K) &= \mathbb{P}^*\left(S_0 \exp\left[\sigma W_T^* + \left(r - \frac{1}{2}\sigma^2\right) T\right] < K\right) \\
&= \mathbb{P}^*\left(W_T^* < \frac{1}{\sigma}\left\{\ln\frac{K}{S_0} - \left(r - \frac{1}{2}\sigma^2\right) T\right\}\right) \\
&= \Phi\left(\frac{1}{\sigma\sqrt{T}}\left\{\ln\frac{K}{S_0} - \left(r - \frac{1}{2}\sigma^2\right) T\right\}\right) \\
&= \Phi(-d_2^0)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}^*(S_T I_{K>S_T}) &= \mathbb{E}^*\left(S_0 \exp\left[\sigma W_T^* + \left(r - \frac{1}{2}\sigma^2\right) T\right] I_{K>S_T}\right) \\
&= S_0 e^{rT} \overline{\mathbb{P}}(S_T < K) \\
&= S_0 e^{rT} \Phi\left(\frac{1}{\sigma\sqrt{T}}\left\{\ln\frac{K}{S_0} - \left(r + \frac{1}{2}\sigma^2\right) T\right\}\right) \\
&= S_0 e^{rT} \Phi(-d_1^0)
\end{aligned}
$$

where $\overline{W}_t = W_t^* - \sigma t$ is Brownian motion under the measure $\overline{\mathbb{P}}$ and the second line follows from the Cameron-Martin-Girsanov Theorem. Similarly:

$$
\begin{aligned}
\mathbb{E}^*(S_T^2 I_{K>S_T}) &= \mathbb{E}^*(S_0^2 e^{2\sigma W_T^* + (2r-\sigma^2)T} I_{K>S_T}) \\
&= S_0^2 e^{(2r+\sigma^2)T} \mathbb{E}^*(e^{2\sigma W_T^* - 2\sigma^2 T} I_{K>S_T}) \\
&= S_0^2 e^{(2r+\sigma^2)T} \tilde{\mathbb{E}}(I_{K>S_T}) \\
&= S_0^2 e^{(2r+\sigma^2)T} \tilde{\mathbb{P}}\left(S_0 \exp\left(\sigma \tilde{W}_T + \left(r + \frac{3}{2}\sigma^2\right) T\right) < K\right) \\
&= S_0^2 e^{(2r+\sigma^2)T} \Phi\left(\frac{1}{\sigma\sqrt{T}}\left\{\ln\frac{K}{S_0} - \left(r + \frac{3}{2}\sigma^2\right) T\right\}\right)
\end{aligned}
$$

where $\tilde{W}_t = W_t^* - 2\sigma t$ is Brownian motion under the measure $\tilde{\mathbb{P}}$.

# Chapter 9

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## Barrier Options


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